For Reference

NOT TO BE TAKEN FROM THIS ROOM

Ex libris
universitatis
albertaensis

QUAECUMQUE VERA

THE UNIVERSITY OF ALBERTA

A ROCKING BLOCK SUBJECTED TO ARTIFICIAL EARTHQUAKES

by

© JEFFREY N. WELLING

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MECHANICAL ENGINEERING

EDMONTON, ALBERTA

FALL, 1971

Digitized by the Internet Archive
in 2023 with funding from
University of Alberta Library

https://archive.org/details/Welling1971

ABSTRACT

For a wide range of block sizes and earthquake magnitude, the probability that a free standing rigid block will survive without overturning when subjected to earthquake ground acceleration is determined. Survival statistics are obtained by computer simulation which consisted of calculating the block's motion when it is subjected to each of a large number of different artificial earthquakes and determining for each earthquake whether the block overturns or not. Two types of artificial earthquake are used and their properties and the method of generation of each are discussed. From comparison of the results obtained with each, it is concluded that the simpler earthquake model, which consists of impulses occurring at random time intervals, can safely be used to estimate survival probabilities.

ACKNOWLEDGEMENTS

The author wishes to thank

- Dr. J.S. Kennedy for his guidance and supervision
- Mr. A.S. Aggarwal for his suggestions and assistance
- Miriam Welling for her patience and encouragement and an excellent job of typing
- The University of Alberta for financial support.

TABLE OF CONTENTS

CHAPTER PAGE

I. INTRODUCTION 1

1.1 Introduction to the Problem 1

1.2 Aims of the Thesis 4

II. ARTIFICIAL EARTHQUAKES 5

2.1 Random Impulse Model Earthquakes 5

2.1.1 Response Spectrum and Spectrum Intensity ... 5

2.1.2 Random Impulse Model of Earthquake 6

2.1.3 Theoretical Response Spectrum 10

2.1.4 Computer Evaluated Response Spectrum 12

2.2 Filtered White Noise Model Earthquakes 20

2.2.1. Definitions and Concepts from Random Process Theory 20

2.2.2 Filtered White Noise Earthquake Simulation . 23

2.2.3 Generation of Filtered White Noise Model Earthquakes 25

2.2.4 Response Spectra of Filtered White Noise ... 30

III. ROCKING BLOCK EXCITED BY RANDOM IMPULSE MODEL EARTHQUAKES 33

3.1 Free Vibration of a Rocking Block 33

3.2 Overturning by Random Impulses as per Housner 37

3.3 Survival Probabilities Determined by Simulation ... 41

3.4 Results of Computer Simulation 47

CHAPTER PAGE

IV ROCKING BLOCK EXCITED BY FILTERED WHITE NOISE MODEL EARTHQUAKE 57

4.1 Introduction 57

4.2 Forced Vibration of a Rocking Block 57

4.3 Computer Simulation of a Rocking Block Subjected to Filtered White Noise 60

4.4 Results of Computer Simulation 66

V CONCLUSIONS and RECOMMENDATIONS 72

5.1 Summary 72

5.2 Conclusions 73

5.3 Recommendations 74

BIBLIOGRAPHY 75

LIST OF TABLES

TABLE PAGE

2.1 Comparison of Theoretical and Computer Evaluated Spectrum Values for Random Impulse Model Earthquake 16

4.1 Corresponding Values of S_{vt}/α and Q/α 68

LIST OF FIGURES

FIGURE PAGE

2.1 Damped Linear Oscillator Subjected to Base Acceleration 7

2.2 Typical Velocity Response Spectra 7

2.3 Average Velocity Response Spectra for Recorded Strong Motion Earthquakes 7

2.4 Average Response Spectra for a Random Impulse Model Earthquake 17

2.5 - 2.7 Average Undamped Velocity Response Spectrum Curves for Random Impulse Model Earthquakes 18

2.8 Theoretical Undamped Response Spectrum Value Versus the Average Ordinate of the Calculated Spectrum for Random Impulse Model Earthquakes ... 19

2.9 Average Response Spectra of Filtered White Noise Model Earthquake Compared With the Spectrum Formula Equation 2.5 33

3.1 Rigid Block in Free Vibration 52

3.2 Rigid Block Subjected Base Acceleration 52

3.3 Survival Curves for Blocks Subjected to Random Impulse Model Earthquakes 53

3.4 Computer Simulation Results Compared to Theoretical Prediction for 50% Survival Rate 54

FIGURE PAGE

3.5 1. Variation in Survival Rates for Repeated Simulations 55

2. The Effect of Energy Loss at Impact 55

3.6 Block Just Prior to Impact 56

4.1 Survival Curves for Blocks Subjected to Filtered White Noise Model Earthquakes 70

4.2 Comparison of Survival Data for the Two Different Earthquake Models 71

NOTATION

E	mechanical energy
$E[x]$	ensemble average of $x(t)$
$F_i \Delta t$	D'Alembert force impulse
$f(t)$	D'Alembert force
$G(i\omega)$	transfer function of a linear differential equation
H	$= +1$ when angular displacement is positive
	$= -1$ when angular displacement is negative
h	height to center of gravity of block
	= one half the height of the block
I	moment of inertia
i	$= \sqrt{-1}$
i, j	subscripts
K	kinetic energy
M	mass of block
N	number of impulses in random impulse model earthquake
$N(t)$	approximate white noise
P	potential energy
Q	scale factor for filtered white noise model earthquakes
q	block size parameter
R	semi-diagonal of block
$R_x(\tau)$	correlation function

R_i	discrete values of filtered white noise acceleration
r	velocity reduction factor due to energy loss at impact
$S(\omega)$	power spectral density
$S_v(\eta,\omega)$	relative velocity response spectrum
$S_{vt}(0,\omega)$	theoretical undamped response spectrum
SI_η	spectrum intensity
t	time
Δt	time interval
T	period of free vibration
v_i	ith impulse in random impulse model earthquake
Δv	impulse magnitude
W	weight of block
w	semi-width of block
X	impulse magnitude
$\bar{x}$	time average of $x(t)$
$\ddot{x}(t)$	base acceleration
$y(t)$	relative displacement of oscillator mass
$\ddot{z}(t)$	filtered white noise
α	slenderness ratio
β,γ	random numbers
$\delta(t)$	Dirac delta function
η	fraction of critical damping
θ	angular displacement of block
τ	time interval

ϕ	$= \frac{\theta}{\alpha}$ non-dimensional displacement parameter
ω	circular frequency

Other symbols are defined when used.

CHAPTER I

INTRODUCTION

1.1 Introduction to the Problem

In designing structures for earthquake resistance it is common practice to apply a static horizontal load in place of the actual inertia load. In the case of tall slender or top heavy structures, it is possible that the overturning moment due to the design seismic load is greater than the opposing moment due to the weight of the structure necessitating special design of foundations and pilings to prevent overturning. However since the actual seismic load is dynamic rather than static and the overturning of such structures seldom occurs during even severe earthquakes, it is of interest to consider the dynamic response of these inverted pendulum type structures to earthquake ground acceleration. Two examples of this type of behaviour that have been reported in the literature are: (a) petroleum cracking towers in California during the Arvin-Tehachapi earthquake of 1952, and (b) golf-ball-on-a-tee water towers during the Chilean earthquakes of 1960. These structures survived with relatively little damage when other more stable appearing structures were severely damaged. It was observed that the cracking towers in California had stretched their anchor bolts which suggested that their behaviour had been that of a rigid body in a rocking motion rather than that of a cantilever beam undergoing elastic vibration.

In 1963 Housner (1)* considered the problem of a rigid block resting on a horizontal base which was accelerated by random impulses. He found that tall slender blocks when subjected to this type of excitation were more stable against overturning than they appeared to be when analyzed under the application of a static horizontal inertia load. Housner also showed that if two geometrically similar blocks of different size were subjected to the same ground motion, the larger block would be more stable than the smaller. In considering the probability of survival of a block subjected to the random impulse excitation, Housner was only able to determine whether this probability would be greater than or less than 50 per cent. That is, he derived a formula relating the size and geometry of the block to the magnitude of the excitation to give a 50 per cent survival rate.

An earlier study of a similar problem was described in a paper by Muto, Umemera and Sonobe (2) in 1960. This paper describes the dynamic testing of models of slender structures resting on a base which was given horizontal sinusoidal motion. It was found that the harmonic motion necessary to overturn structures would have to be of very long periods and of large amplitudes. Such motions do not occur in earthquakes and it was concluded that tall slender structures would not be as easily overturned by earthquakes as might be supposed.

---

* Numbers in parentheses refer to references in Bibliography

To determine the probability of a rocking structure overturning under earthquake action is an example of the first passage time problem of random vibration theory. There exists no exact theoretical solution to determine the probability that a random process will exceed specified limits for the first time in a given time interval. Several approximate solutions have been derived but to apply them to a particular process it is necessary to know certain statistical propertiesof that process (14,15,16). If the random process that is being considered is the response of a linear time invariant system, it is possible to determine the required response statistics from the excitation statistics and the impulse response function of the differential equation by applying superposition techniques via the Duhamel integral.

It will be seen that the differential equation of motion for the case of a rocking block is not continuous but that it is step-wise linear so that superposition cannot be used. This makes it impossible to calculate the statistical properties of the angular displacement of the block from those of the excitation using standard analytical techniques. In such cases response statistics can be estimated from data obtained from repeated observations of the system. With the speed of modern computers, it is possible to obtain a large number of observations by simulating the system on a digital computer. This is the procedure that is used in the investigation described in this thesis. The details of a block's motion are computed to determine whether or not it overturns when it is subjected in turn to each of a large number of different earthquakes of the same

magnitude. From the number of earthquakes that the block survives without overturning, the probability of that block surviving an earthquake of that magnitude can be determined.

There does not exist a large number of records of actual earthquake ground motion of a given magnitude that could be used as the excitation in such a procedure. It is necessary to generate random functions suitable for use as models of earthquake ground motion. The two types of earthquake models used in this investigation were: (a) ground acceleration consisting of impulses which are random in time, and (b) ground acceleration defined by a segmentally linear function consisting of randomly varying discrete values at small time intervals Δt connected by straight lines.

1.2 Aims of the Thesis

1. To confirm and extend the results obtained by G. Housner (1) concerning the probability of survival of a rocking block subjected to earthquake acceleration modeled as a series of impulses arriving at random time intervals.

2. To obtain similar results using a more comprehensive model of earthquake acceleration obtained as a form of filtered white noise and compare with those obtained above.

3. To use simulation techniques, modeling the system on the I.B.M. 360/67 digital computer at the University of Alberta Computing Center, to determine the required statistics.

CHAPTER II

ARTIFICIAL EARTHQUAKES

2.1 Random Impulse Model Earthquakes

2.1.1 Response Spectrum and Spectrum Intensity

A concept which has been developed to investigate and delineate the character of earthquake ground motion and to aid in determining structural response to earthquakes is the response spectrum. The response spectrum of an earthquake is the curve obtained when, for a series of linear oscillators whose undamped natural periods of vibration range from .10 seconds to 3.00 seconds, the maximum response of each oscillator is plotted against its period for a particular damping factor. The response function which is commonly used is the relative velocity. Consider the spring-mass-damper system shown in Figure 2.1 which is excited by the base acceleration $\ddot{x}(t)$ and experiences the displacement $y(t)$ of the mass relative to the base. The equations determining the relative velocity response spectrum value, $S_v(\eta,\omega)$, are:

$$\ddot{y} + 2\eta\omega\dot{y} + \omega^2 y = -\ddot{x}(t)$$

$$S_v(\eta,\omega) = |\dot{y}|\text{max.} \qquad 2.1$$

in which η is the fraction of critical damping and ω denotes the natural circular frequency. Repeated calculations of S_v for different natural frequencies and different damping factors for a given acceleration record $\ddot{x}(t)$ produce a set of curves such as are shown in Figure 2.2 for one component of the Taft, Calif. earthquake of 1952, (1).

Figure 2.3 shows the response spectra curves obtained when the spectrum values from eight actual earthquakes are averaged. Though the response spectrum can be used to determine the expected maximum response of linear structures to a given excitation, for purposes of the present investigation, average spectra will be used as the means to compare artificially generated earthquakes to determine their suitability.

Because the structures in an area which is experiencing an earthquake will have various natural frequencies, it is desirable to have some average over all pertinent frequencies which would be an indication of the severity of the earthquake. A single parameter which accomplishes this is the Spectrum Intensity, SI_η, which is defined as the area under the velocity response spectrum with damping η between the periods .10 seconds and 2.50 seconds.

To form the average velocity spectra of the actual earthquakes (Figure 2.3), the eight individual spectra were first normalized so that the undamped spectrum intensities were all the same and then averaged.

2.1.2 Random Impulse Model of Earthquake

One of the first attempts to explain earthquake accelerations in terms of a random process was made by Housner in 1947 (7) when he showed the similarity between response spectra for actual earthquakes and response spectra for accelerations consisting of impulses arriving randomly in time.

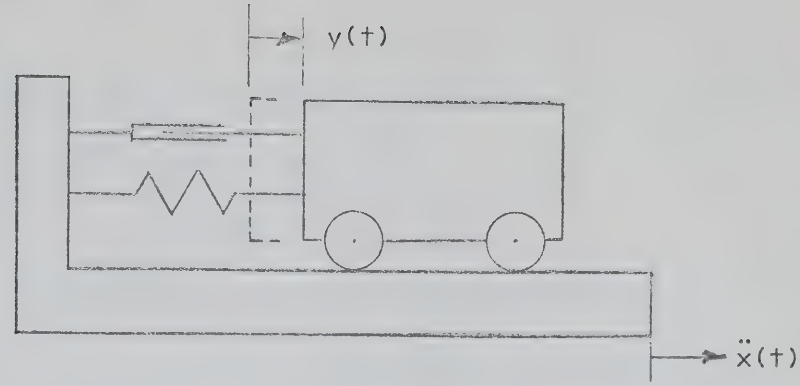

Figure 2.1 Damped Linear Oscillator Subjected To Base Acceleration.

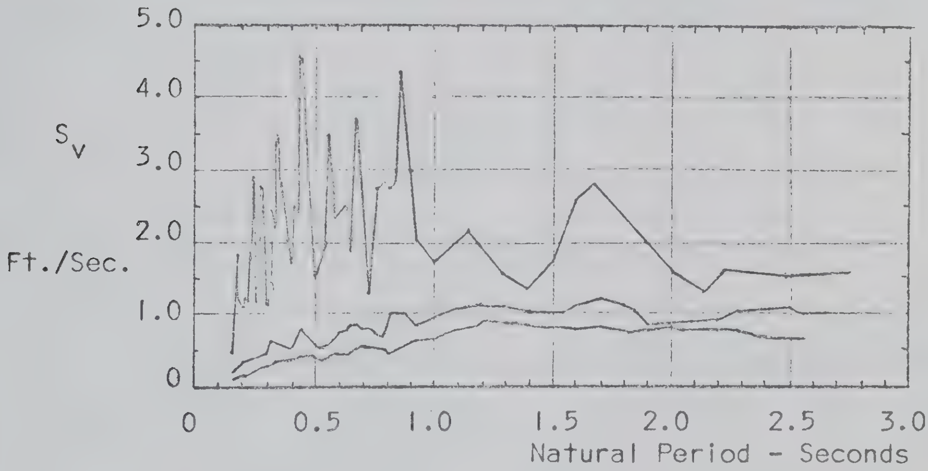

Figure 2.2 Typical Velocity Response Spectra.

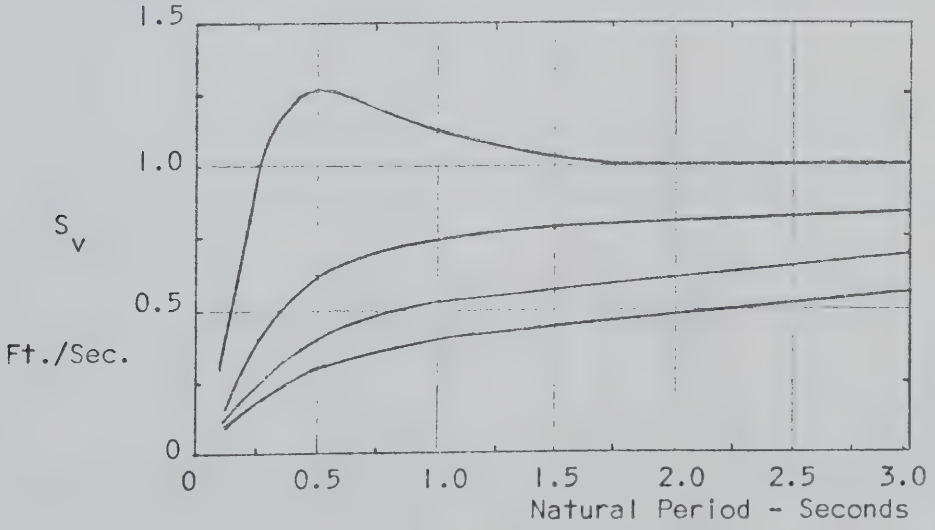

Figure 2.3 Average Velocity Response Spectra For Recorded Strong Motion Earthquakes.

Consider again the linear system shown in Figure 2.1 and the corresponding differential equation:

$$\ddot{y} + 2\eta\omega\dot{y} + \omega^2 y = -\ddot{x}(t)$$

Let the base acceleration consist of an impulse of magnitude X occurring at time $t = 0$, $\ddot{x}(t) = X\delta(t)$, where X is a real constant and $\delta(t)$ is the Dirac delta function.

Considering the base, such an acceleration impulse is equivalent to a step change in the base velocity. This may be shown as follows:

$$\ddot{x} = \frac{d\dot{x}}{dt} = X\delta(t)$$

$$\Delta\dot{x} = \int_{\dot{x}}^{\dot{x}+\Delta\dot{x}} d\dot{x} = \int_{-\varepsilon}^{+\varepsilon} X\delta(t)dt = X$$

Considering the mass, m, the base acceleration may be replaced by a D'Alembert force:

$$f(t) = -m\ddot{x}(t) = -mX\delta(t)$$

From impulse -momentum considerations

$$f(t)dt = m\ d\dot{y}$$

The mass m thus experiences a step change in its relative velocity due to the impulse in the base acceleration as may be shown:

$$\Delta\dot{y} = \int_{\dot{y}}^{\dot{y}+\Delta\dot{y}} d\dot{y} = \int_{-\varepsilon}^{+\varepsilon} \frac{f(t)}{m}\ dt = \int_{-\varepsilon}^{+\varepsilon} -X\delta(t)dt = -X$$

Thus an acceleration impulse of magnitude X which causes a step change in the velocity of the base is experienced by the mass as a step change in velocity of magnitude X. If the system

were at rest prior to the impulse, the effect of the impulse would be the same as free vibration of the mass with initial conditions $y(0) = 0$ and $\dot{y}(0) = -X$ for which the solution for $y(t)$, the relative displacement, is

$$y(t) = -\frac{x}{\omega} e^{-\eta\omega t} \sin \omega t$$

For a base acceleration consisting of a sequence of N impulses of magnitude v_i occurring at random times t_i where $i = 1,2,\ldots.N$, the resulting motion could be found by superposition to be:

$$y = \sum_{i=1}^{N} \frac{1}{\omega} v_i \, e^{-\eta\omega(t-t_i)} \sin \omega(t-t_i) \qquad 2.2$$

Model earthquakes composed of random ground acceleration impulses or, what is equivalent, random step changes in ground velocity were produced on the computer by using a subroutine for generating pseudo-random numbers (17). The duration of the model earthquakes was chosen to be 30 seconds which has been estimated to be the approximate duration of the strong-motion phase of a very large earthquake (10). It was assumed that 100 impulses occurring at random time intervals for a total elapsed time of approximately 30 seconds would be a suitable model to simulate the earthquake ground accelerations. Thus the mean time interval between impulses is required to be 0.30 seconds. This was obtained by selecting the time interval as a random number from a uniform distribution between 0 and 0.60. The impulses were taken to be of constant magnitude, Δv, but random in

sign. To determine the sign (direction) of the impulse, a random number was selected from a uniform distribution between -1.0 and +1.0 and the sign of the impulse set the same as the sign of the random number. In this way there is equal probability of any given impulse being positive or negative. Thus 100 times, t_i, separated by intervals of random length and 100 impulses, $v_i = \pm \Delta v$, were selected to define the model earthquake.

2.1.3 Theoretical Response Spectrum

To determine the theoretical undamped velocity response spectrum value, denoted $S_{vt}(0,\omega)$, for this excitation according to Equation 2.1, we must consider the maximum value to $|\dot{y}|$ for $\eta=0$. From Equation 2.2 with $\eta=0$

$$y = \sum_{i=1}^{N} \frac{1}{\omega} v_i \sin \omega(t-t_i)$$

Differentiation yields

$$\begin{aligned}\dot{y} &= \sum_{i=1}^{N} v_i \cos \omega(t-t_i) \\ &= \sum_{i=1}^{N} v_i (\cos \omega t \cos \omega t_i + \sin \omega t \sin \omega t_i) \\ &= \left(\sum_{i=1}^{N} v_i \cos \omega t_i\right) \cos \omega t \\ &\qquad + \left(\sum_{i=1}^{N} v_i \sin \omega t_i\right) \sin \omega t\end{aligned}$$

The addition of these two harmonic functions results in a single function with amplitude:

$$|\dot{y}|_{max} = \sqrt{(\sum_{i=1}^{N} v_i \cos \omega t_i)^2 + (\sum_{i=1}^{N} v_i \sin \omega t_i)^2}$$

Substitution into Equation 2.1 and expanding yields for the theoretical undamped spectrum value corresponding to natural frequency ω:

$$S_{vt}\ (0,\omega) = \sqrt{\sum_{i=1}^{N} v_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} v_i v_j \cos \omega(t_i - t_j)}$$

But $v_i = \pm \Delta v$ so the first summation becomes

$$\sum_{i=1}^{N} v_i^2 = N(\Delta v)^2$$

In the double summation, the product $v_i v_j$ has equal probability of being either $+(\Delta v)^2$ or $-(\Delta v)^2$. Due to the randomness of the times t_i the cosine will have equal probability of being any value between -1 and +1 so that each term in the double summation has equal probability of being any number between $+(\Delta v)^2$ and $-(\Delta v)^2$. With the large number of terms involved (99 + 98 + ...+2 +1 for N = 100), the double summation would be equal to zero and the spectrum value becomes:

$$\begin{aligned} S_{vt}\ (0,\omega) &= \sqrt{N(\Delta v)^2} \\ &= \sqrt{N}\ \Delta v \\ &= 10\ \Delta v \end{aligned} \qquad 2.3$$

for the case of 100 impulses.

Since the spectrum value is independent of ω, the undamped response spectrum curve is a straight line. When this model of earthquake acceleration was first considered (7) the average zero damped spectrum curves of actual earth-

quakes that were available for comparison had been calculated by a mechanical analogue that incorporated approximately one per cent of critical damping. This was sufficient to eliminate the hump from the zero damped spectrum so that the straight line spectrum of the random impulse type acceleration compared favourably with the actual earthquake results and it was assumed that this type of ground acceleration could approximate earthquake action.

For vibration periods in excess of 1.5 seconds, the average undamped spectrum as shown in Figure 2.3 becomes a flat curve independent of period so that for structures with large natural periods, the random impulse type of excitation could be a good model for earthquake ground motion.

2.1.4 Computer Evaluated Response Spectrum

To determine the actual velocity response spectrum curves for a random impulse model earthquake generated as outlined above, the linear system of Figure 2.1 was simulated on the computer. The simulation consisted of calculating the details of the motion of the oscillator mass due to the base acceleration $\ddot{x}(t)$ defined by the earthquake's impulses v_i at times t_i. By repeating the simulation for oscillators with different natural periods and different damping and determining the maximum relative velocity attained in each case, the necessary data to plot the spectrum curves

was determined.

In section 2.1.2 it was shown that the only effect of a ground impulse was to cause a step change in the velocity of the oscillator so that between impulses it moves as free vibration. Thus the motion of the oscillator was calculated as a series of free vibration problems defined as:

$$\ddot{y}_i(t) + 2\eta\omega\dot{y}(t) + \omega^2 y_i(t) = 0$$

for $$0 \leq t \leq \Delta t_i$$

with $$y_i(0) = y_{i-1}(\Delta t_{i-1})$$

and $$\dot{y}_i(0) = \dot{y}_{i-1}(\Delta t_{i-1}) + v_i$$

where the subscript $i = 1,2,\ldots\ldots 100$ denotes the time interval in which the motion is being calculated (i.e. the interval between t_i and t_{i+1}) and $\Delta t_i = t_{i+1} - t_i$ is the length of this time interval.

The motion of the oscillator is started by the first impulse, v_1, of the earthquake so that the initial conditions for the first interval were $y_1(0) = 0$ and $\dot{y}_1(0) = v_1$.

The displacement and velocity were determined at the end of each interval to be used for the initial conditions of the next interval. The velocity was also calculated at all times when the value of the displacement was zero.

Whenever there was a zero crossing within an interval, a new initial condition problem would be considered in the sub-

interval from the time at which $y = 0$ to the end of the interval. The initial conditions for this sub-interval would be $y(0) = 0$ and $\dot{y}(0)$ = velocity at the time of the zero crossing.

For small damping the general solution for the displacement in all cases was taken as:

$$y(t) = Y\ e^{-\eta\omega t} \sin\ (\omega t + \lambda)$$

with the constants Y and λ determined from the initial conditions.

From all the values of velocity calculated during the oscillator's motion, the maximum velocity is determined and this is the velocity response spectrum value for the particular value of natural period and damping factor being considered. The response spectrum curve is formed by plotting the spectrum values for natural periods of .10, .20, .30... 2.50 seconds and joining them by straight lines. Because of the random nature of the excitation the response spectrum for any one earthquake is not as significant as that obtained as the average for a large number of earthquakes. Average spectrum curves were obtained by averaging in each case the results obtained in analyzing 150 random impulse model earthquakes with a given impulse magnitude.

The average response spectra obtained for model earthquakes with impulse magnitude $\Delta v = 0.25$ and damping ratios of 0, 0.02, 0.05, and 0.10 are shown in Figure 2.4. Typical undamped average spectrum curves are shown in Figures 2.5, 2.6, and 2.7. Also shown in these figures are the

average ordinates of the calculated spectrum curve and the corresponding straight line spectrum, S_{vt}, predicted by theoretical considerations. Table 2.1 summarizes the results obtained in the calculation of average undamped velocity spectra of random impulse model earthquakes with impulse magnitudes ranging from 0.125 ft./sec. to 0.35 ft./sec.

The spectrum intensities were calculated as the area under the spectrum curve and the average ordinate determined as that value which a straight line spectrum must have in order to have the same spectrum intensity.

In comparing the calculated and theoretical undamped spectrums, it was found that the calculated spectrum was independent of the period of vibration as predicted, but that the average ordinate of the calculated spectrum was consistently higher than the predicted spectrum value,

$$S_{vt}(0,\omega) = \sqrt{N}\,\Delta v$$

for all impulse magnitudes considered. When corresponding values of the average ordinate of the calculated spectrum and the theoretical spectrum value were plotted as shown in Figure 2.8., it was found that there was a linear relationship between the two characterized by the slope:

$$\frac{\text{average ordinate}}{S_{vt}} = \frac{8}{7}$$

A corrected expression for S_v in terms of the number of impulses and the impulse magnitude which would

agree with the average ordinate of the calculated spectrum would be

$$S_v(0,\omega) = \frac{8}{7}\sqrt{N}\ \ \Delta v$$

Table 2.1

Comparison of Theoretical and Computer Evaluated Spectrum Values For Random Impulse Model Earthquakes.

Impulse Magnitude	Theoretical Spectrum value $S_{vt}(0,\omega)=\sqrt{N}\cdot\Delta v$	Calculated Spectrum	
		Average Ordinate	Spectrum Intensity
.125	1.25	1.40	3.36
.150	1.50	1.68	4.03
.187	1.87	2.10	5.04
.200	2.00	2.24	5.39
.250	2.50	2.82	6.78
.350	3.50	3.96	9.50

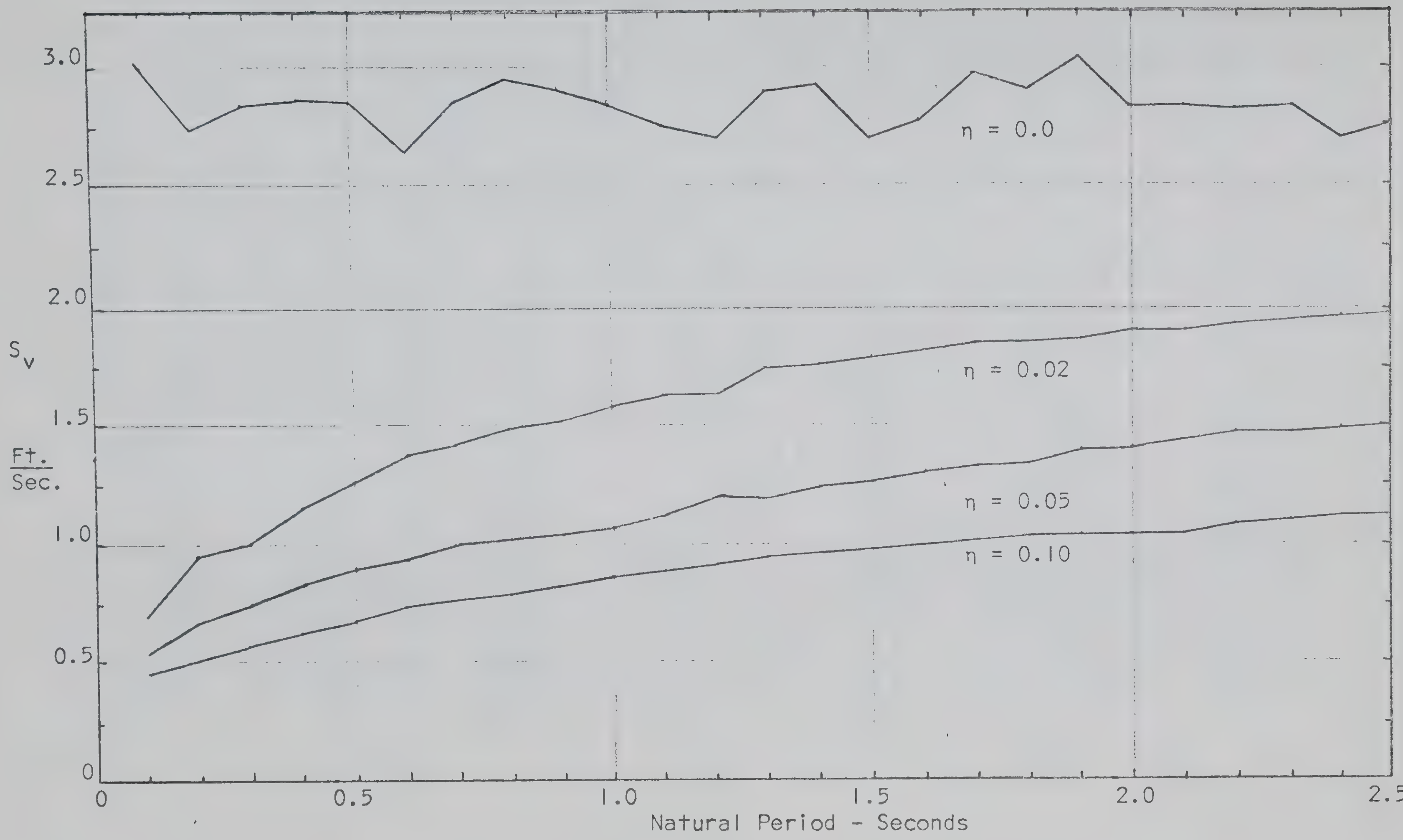

Figure 2.4 Average Response Spectra For A Random Impulse Model Earthquake.

Figures 2.5 - 2.7 Average Undamped Velocity Response Spectrum Curves For Random Impulse Model Earthquakes.

╲╱╲ Spectrum Curve

- - - - - - - Average Ordinate of Spectrum Curve

—•—•— Theoretical Spectrum Value, $S_{vt}(0,\omega)$

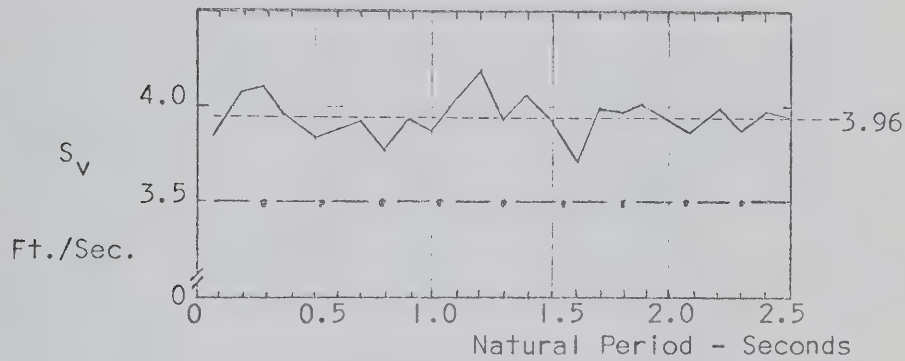

Figure 2.5 $S_{vt} = 3.50$

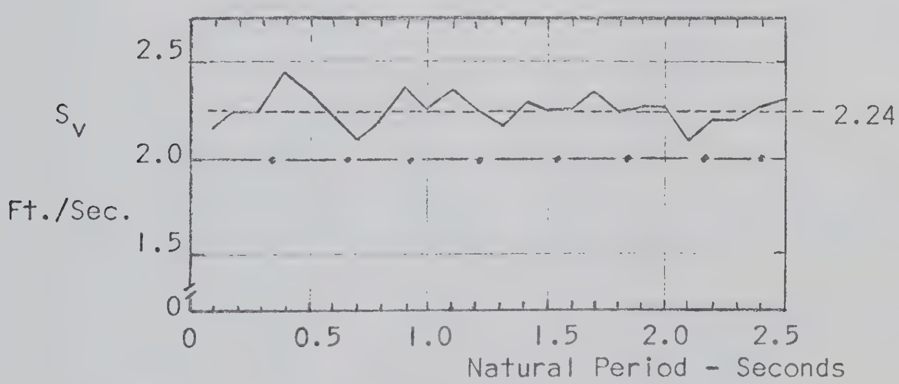

Figure 2.6 $S_{vt} = 2.00$

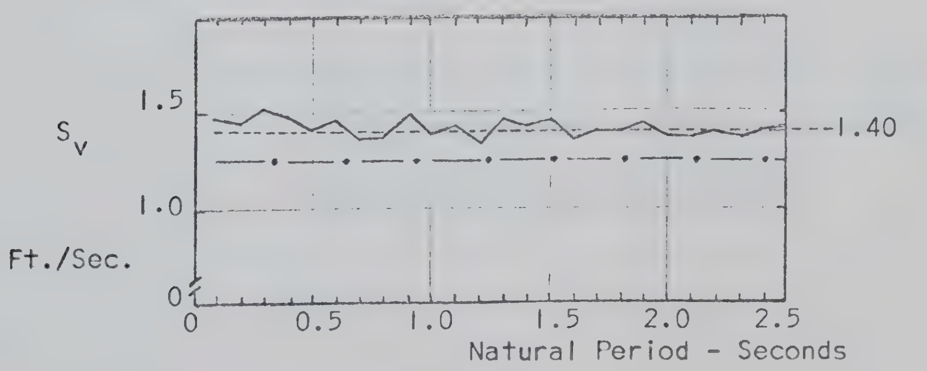

Figure 2.7 $S_{vt} = 1.25$

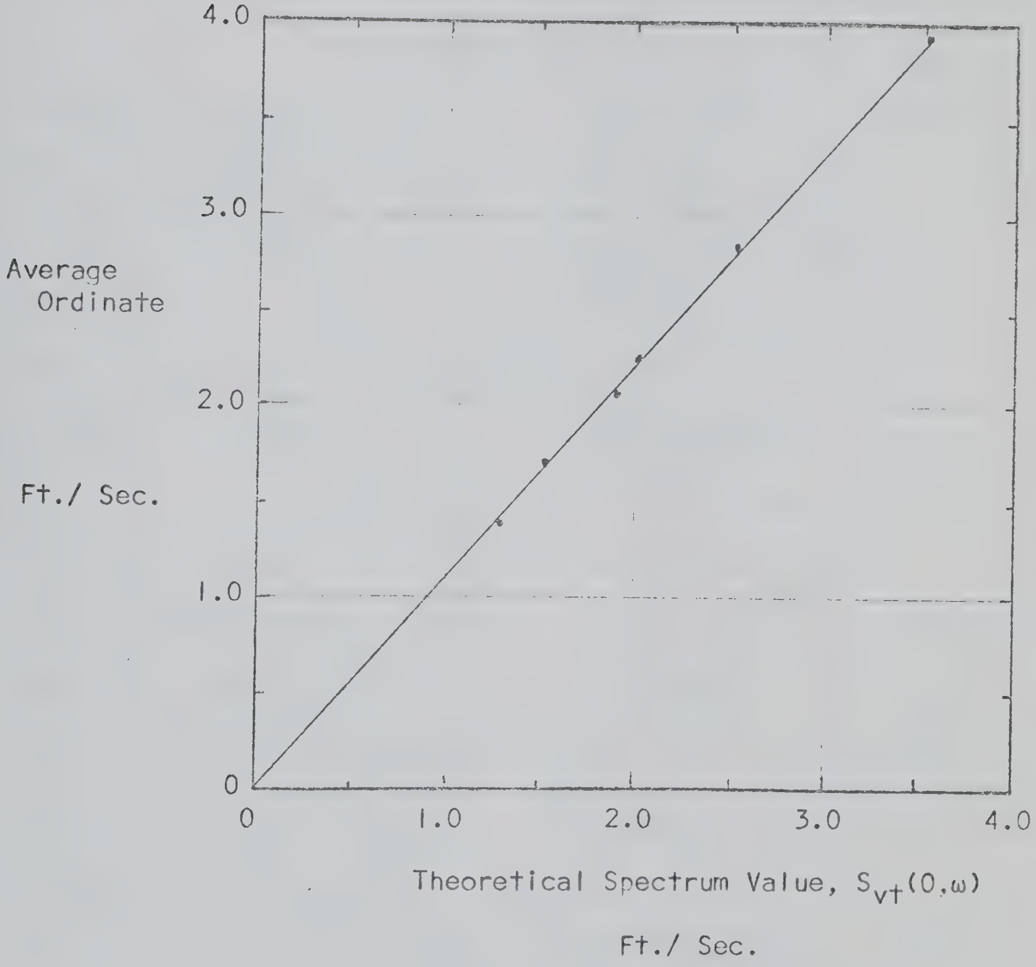

Figure 2.8 Theoretical Undamped Response Spectrum Value Versus the Average Ordinate of the Calculated Spectrum for Random Impulse Model Earthquakes.

2.2 Filtered White Noise Model Earthquakes

2.2.1 Definitions and Concepts From Random Process Theory

A random process $x(t)$ is one which cannot be described by a single record, for as the process is repeated, the records obtained continually differ from each other. It is necessary to consider the entire ensemble of possible time histories $x_1(t)$, $x_2(t)$,......$x_i(t)$ from which a statistical description of the process can be derived. The first order probability distribution describes the distribution of values $x(t_1)$ which occur in the ensemble for $t=t_1$ by means of a graph of the first order probability density function $p_1(x,t_1)$. The fraction of ensemble members for which $x(t_1)$ lies between x and x + dx, which is equivalent to the probability of finding $x(t_1)$ between x and x + dx in any given record of the random process, is given by $p_1(x,t_1)dx$. The expected or mean value of $x(t_1)$ is given by $E[x(t_1)] = \int_{-\infty}^{\infty} xp_1(x,t_1)\,dx$ where E[ ] indicates an average taken across the ensemble. Higher order probability distributions describe the distribution of of products $x(t_1)x(t_2)$, $x(t_1)x(t_2)x(t_3)$, etc. across the ensemble and provide a measure of the mutual relation between the values of the random function at different times. Sufficient probability information for most processes can be determined from the first and second order probability distributions. The second order probability density function

is denoted $p_2(x_1,t_1;\ x_2,t_2)$ and $p_2(x_1,t_1;x_2,t_2)dx_1dx_2$ is the probability of finding any one of the sample functions $x_i(t)$ in the interval (x_1,x_1+dx_1) at time $t=t_1$ and in the interval $(x_2,\ x_2 + dx_2)$ at time $t = t_2$.

A stationary random process is one for which the statistical parameters do not vary with time. For such a process the first order probability density function is time invariant, $p_1(x,t_1)$ becomes $p_1(x)$ and the expected value of x at any time is $E[x] = \int_{-\infty}^{\infty} xp_1(x)dx$. The interdependence of of the values of x at two instants of time depends only on the time interval between them, $\tau = t_2 - t_1$, so that the second order probability density function is a function of τ, $p_2(x_1,\ x_2,\ \tau)$ Because the parameters do not vary with time it becomes reasonable to consider averages taken with respect to time. According to the ergodic assumption for a stationary random process, statistical properties determined by averaging over time for any one sample will be equal to those determined by averaging across the ensemble. For a given sample function $x_i(t)$ the mean value of x as determined by the time average, denoted $\bar{x}$, is:

$$\bar{x} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

By the ergodic assumption every sample function will have the same average which will be equal to the average across the ensemble: $\bar{x} = E[x]$

The variance, σ^2, is the ensemble average of the square of the deviation from the mean:

$$\sigma^2 = E[(x - E[x])^2] = \int_{-\infty}^{\infty} (x - E[x])^2 \, p_1(x)dx$$

For processes with mean zero, i.e. $E[x]=0$, the variance is identical with the mean square: $\sigma^2 = E[x^2]$.

A statistical measure involving second order probability densities is the correlation function defined as the average of the product $x(t)x(t+\tau)$ and denoted $R_x(\tau)$.

$$R_x(\tau) = E[x_1(t)x_2(t+\tau)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2, \tau) \, dx_1 \, dx_2$$

In terms of the time averages the correlation function is defined as

$$R_x(\tau) = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau)dt$$

A frequency analysis of a stationary random process can be accomplished by considering the power spectral density, $S(\omega)$, which is defined in terms of the correlation function by using the Fourier integral transform:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau} \, d\tau$$

For a stationary random process $S(\omega)d\omega$ is a measure of the average power available in the frequency interval ω to $\omega + d\omega$. Whereas the response spectrum discussed earlier (Section 2.1) is a concept used only in earthquake engineering, power spectral density is used more generally as a means to describe the frequency content of a stationary

random process. A random process for which the power spectral density has a constant value over the entire range of frequencies, i.e. $S(\omega)$ constant for all ω, is referred to as white noise.

A normal or Gaussian random process is one for which the first order probability density function is a normal curve and all higher order probability densities are similarly functions of known form which can be defined in terms of the power spectral density. In this paper we will be concerned with Gaussian random processes with zero means and the response of a linear system to this type of random excitation.

When a linear time invariant system is excited by a zero mean Gaussian random process, the response will also be a zero mean Gaussian process. The power spectral densities of the excitation and response functions are denoted $S_{in}(\omega)$ and $S_{out}(\omega)$ respectively and are related by

$$S_{out}(\omega) = |G(i\omega)|^2 \; S_{in}(\omega) \qquad 2.4$$

where $G(i\omega)$ is the complex frequency response or transfer function of the linear system. $G(i\omega)$ is a function of frequency such that when the excitation input to the system is $e^{i\omega t}$ the response is $G(i\omega)e^{i\omega t}$.

2.2.2. Filtered White Noise Earthquake Similation

The relation between the excitation and response spectral densities (Equation 2.4) forms the basis for the method used by Housner and Jennings (10) to produce Gaussian stationary random functions with a specified spectral

density suitable for use as models of earthquake ground acceleration. If a linear spring-mass-damper system is excited by white noise so that $S_{in}(\omega)$ = constant, then the magnitude and form of the spectral density of the response will depend on the transfer function of the system. By proper selection of the resonance and damping characteristics, the system can be designed to produce a random motion with the desired statistical properties. The output obtained from such a system is referred to as filtered white noise.

Using a linear system such as that shown in Figure 2.1 as the filtering system, with the base acceleration, $\ddot{x}(t)$, being the white noise excitation and the absolute acceleration of the mass, $\ddot{z}(t)$, being the filtered white noise response, Housner and Jennings produced an ensemble of eight pseudo-earthquake accelerograms which they proposed to use in studies of the earthquake response of structures. The same approach was used in the present investigation to produce an ensemble of fifty model earthquakes.

For the filtered white noise to be used as earthquake acceleration, it must have the same spectral density as the recorded ground accelerations of actual earthquakes. However, the statistical properties of earthquakes are known in terms of the average velocity response spectra shown in Figure 2.3. In their paper, Housner and Jennings show that the relation between the velocity response spectra, $S_v(n,\omega,t)$,

and the power spectral density, $S(\omega)$, of recorded earthquakes is given by:

$$S_v(\eta,\omega,t) = 1.796\sqrt{\frac{\pi S(\omega)}{2\eta\omega}\cdot(1 - e^{-2\eta\omega t/2.44})} \qquad 2.5$$

where t is the time duration of the earthquake ensemble. For zero damping this reduces to

$$S_v(0,\omega,t) = 1.150\sqrt{\pi S(\omega)t}$$

When the above relation was used to determine the spectral density to correspond to the average response spectra, it was found that a simple algebraic function of a form which is convenient for linearly filtered white noise could be used for $S(\omega)$:

$$S(\omega) = \frac{0.01238\left(1 + \frac{\omega^2}{147.8}\right)}{\left(1 - \frac{\omega^2}{242}\right)^2 + \frac{\omega^2}{147.8}} \qquad 2.6$$

2.2.3. Generation of Filtered White Noise Model Earthquakes

Consider again a linear mass-spring-damper system as shown in Figure 2.1. The equation of motion can be written as:

$$\ddot{y} + 2bc\dot{y} + c^2y = -\ddot{x}(t)$$

where y is the displacement of the mass relative to the base, $\ddot{x}$ is the forced acceleration of the base and b and c are system constants. If z is defined as the absolute displacement of the mass so that $z = y + x$ then this equation may be written as:

$$\ddot{z} + 2bc\dot{z} + c^2z = 2bc\dot{x} + c^2x \qquad 2.7$$

To determine the transfer function, $G(i\omega)$, between x and z (which is also the transfer function between $\ddot{x}$ and $\ddot{z}$) we set $x = e^{i\omega t}$ and $z = G(i\omega)e^{i\omega t}$ in Equation 2.6. Solving for $G(i\omega)$ yields:

$$G(i\omega) = \frac{1 + i\frac{2b\omega}{c}}{1 - \frac{\omega^2}{c^2} + i\frac{2b\omega}{c}}$$

If the base acceleration $\ddot{x}(t)$ is white noise its power spectral density will be constant, say $S_{in}(\omega) = a$ and the absolute acceleration of the mass, $\ddot{z}(t)$, will be filtered white noise with a spectral density $S_{out}(\omega)$ determined by Equation 2.3 to be:

$$\begin{aligned} S_{out}(\omega) &= |G(i\omega)|^2 \cdot a \\ &= \frac{1 + i\frac{2b\omega}{c}}{1 - \frac{\omega^2}{c^2} + i\frac{2b\omega}{c}} \cdot \frac{1 - i\frac{2b\omega}{c}}{1 - \frac{\omega^2}{c^2} - i\frac{2b\omega}{c}} \cdot a \\ &= \frac{a(1 + 4\frac{b^2}{c^2}\omega^2)}{(1 - \frac{\omega^2}{c^2})^2 + 4\frac{b^2}{c^2}\omega^2} \qquad 2.8 \end{aligned}$$

If the filtered white noise obtained from such a system is to be used as a model for earthquake ground acceleration, it must have a spectral density as given by Equation 2.6. Setting $S_{out}(\omega)$ from Equation 2.8 equal to $S(\omega)$ from Equation 2.6 the required values of the coefficients of the differential equation of the filtering system and the magnitude of the input spectral density were found to be:

$$c^2 = 242, \quad b^2 = 0.410, \quad a = 0.01238 = S_{in}(\omega) .$$

To use this system to produce model earthquakes, white noise was required for use as the excitation. An approximate form of Gaussian white noise, denoted by N(t), was formed by generating a sequence of Gaussian random numbers (17) which were used as the ordinates of the function separated by time intervals Δt and connected by straight lines to form a segmentally linear function. It was shown by Housner and Jennings (10) that if Δt is taken as 0.025 sec. and the random numbers have a mean of zero and a variance $\sigma^2 = \frac{\pi}{\Delta t}$., then the function N(t) will have a spectral density close to unity, $S_N(\omega) = 1.0 \pm .05$, for the range of frequencies of concern. However, it was required that the input acceleration $\ddot{x}(t)$ have a spectral density $S_{in}(\omega) = a$. As the spectral density is a second order function of the process, the acceleration was taken as $\ddot{x}(t) = \sqrt{a}\ N(t)$ to yield the spectral density of $\ddot{x}$ (t) as

$$S_{in}(\omega) = (\sqrt{a})^2 \cdot S_N(\omega) = a$$

as required.

Consider again the differential equation of the filtering system in terms of the relative displacement y,

$$\ddot{y} + 2bc\dot{y} + c^2y = -\ddot{x} = -\sqrt{a}N(t) \qquad 2.9$$

To determine the time record of the absolute acceleration of the mass, $\ddot{z}$ (t), which for the values of the coefficients

determined above will be the required filtered white noise accelerogram for a model earthquake, this equation must be integrated to evaluate

$$\ddot{z}(t) = \ddot{y} + \ddot{x} = -2bc\dot{y} - c^2y.$$

To start the integration initial values of the relative displacement y(0) and velocity $\dot{y}(0)$ are required. Because the excitation was assumed to be a stationary random process, y(t) would also be stationary and it was sufficient to choose the initial values from the probability distributions of y(t) and $\dot{y}(t)$. It can be shown that their variances are

$$E.[y^2(t)] = \frac{\pi a}{4bc^3}$$

$$E\ [\dot{y}^2(t)] = \frac{\pi a}{4bc}$$

If β and γ are two numbers taken at random from a Gaussian distribution with mean zero and variance unity then setting

$$y(0) = \frac{1}{2c}\sqrt{\frac{a\pi}{bc}}\ .\ \beta$$

and

$$\dot{y}(0) = \frac{1}{2}\sqrt{\frac{a\pi}{bc}}\ .\ \gamma$$

will ensure that these initial values are members of Gaussian processes with mean zero and the required variances.

The integration of Equation 2.9 was accomplished on the digital computer using a fourth order Runge-Kutta method. The value of $\ddot{z}$ (t) was determined at 0.025 second intervals and these values joined by straight lines to produce the model

earthquake accelerogram. To obtain accelerograms of thirty seconds duration, 1200 random numbers were required to produce the white noise excitation and similarly 1200 numbers were determined to define $\ddot{z}(t)$.

By repeating the integration, each time with a different sample of the white noise excitation, as many different accelerograms as required can be produced. However, because the computer time required was considerable, only fifty accelerograms were produced to serve as the excitation ensemble in the present investigation.

The filtered white noise accelerograms were modified slightly to enhance their suitability as realistic models for earthquake ground acceleration by ensuring that the ground velocity tends to zero at the end of the earthquake. To achieve this, a corrected base line defined by the function $C_0 + C_1t + C_2t^2$ was superimposed on the accelerogram and the velocity of the ground found by integrating the acceleration as measured from this corrected base line starting with zero initial conditions on the velocity and displacement. The constants C_0, C_1, and C_2 were determined so as to minimize the mean square of the velocity and then the acceleration ordinates were determined with respect to the corrected base line to form the model earthquake accelerograms. The integration to determine the ground velocity is an easy and

exact calculation because of the accelerogram being segmentally linear.

2.2.4 Response Spectra of Filtered White Noise

The relative velocity response spectra of the model earthquakes were determined by considering again a linear oscillator whose base is excited by the adjusted accelerograms which will be denoted $\ddot{z}(t)$. The equation determining the spectrum value for a particular frequency ω and damping ratio η are:

$$\ddot{y} + 2\eta\omega\dot{y} + \omega^2 y = -\ddot{z}(t)$$

$$S_v(\eta,\omega) = |\dot{y}|_{max}$$

The integration of this equation was carried out using a Runge-Kutta method starting with zero initial conditions on the relative displacement and velocity. The integration interval used was half the basic interval of the excitation so that the relative velocity of the oscillator was determined at .0125 second intervals and the calculated velocity with the largest magnitude was taken as the response spectrum value. For each model earthquake spectrum values corresponding to oscillator periods of 0.10, 0.20, ...1.00, 1.20, 1.40 ....2.80 and 3.00 seconds and damping ratios of 0, 0.02, 0.05 and 0.10 were determined to completely specify response spectra and the spectrum intensities. From the spectrum values for each of the fifty model earthquakes,

average spectrum values for the ensemble were determined to define the average response spectrum curves.

To judge the suitability of the model earthquakes, it is preferable to compare their average response spectra with average spectra of recorded earthquakes. However, the model earthquake spectra cannot be compared directly with the recorded earthquake spectra given in Figure 2.3 because they are not of the same duration and in Equation 2.5 it is shown that the spectrum values depend on the duration. The average duration for the recorded earthquakes was 18.6 seconds whereas the duration for the pseudo-earthquakes was 30 seconds. Equation 2.5 can be used to determine the expected average response spectra for the model earthquakes.

Figure 2.9 shows plots of the response spectra formula and the average response spectra of the model earthquakes. The agreement between the two is better than that obtained by Housner and Jennings with their ensemble of eight model earthquakes as would be expected due to the larger ensemble. Such agreement speaks well for the consistency of this method of producing pseudo-earthquakes as well as showing that the accelerograms produced should prove satisfactory as model earthquakes.

The value of the zero damped spectrum intensity, SI_0, was determined for each of the fifty model earthquakes. The values obtained ranged from $SI_0 = 2.68$ to $SI_0 = 3.54$.

The average for the ensemble was $\overline{SI}_0 = 3.13$ and this value was taken as the parameter to indicate the strength of the ensemble. To use the ensemble to represent earthquakes of various strengths, the acceleration values defining the fifty earthquakes were multiplied by a scale factor, say Q, to produce an ensemble with average spectrum intensity $\overline{SI}_0 = 3.13 \cdot Q$.

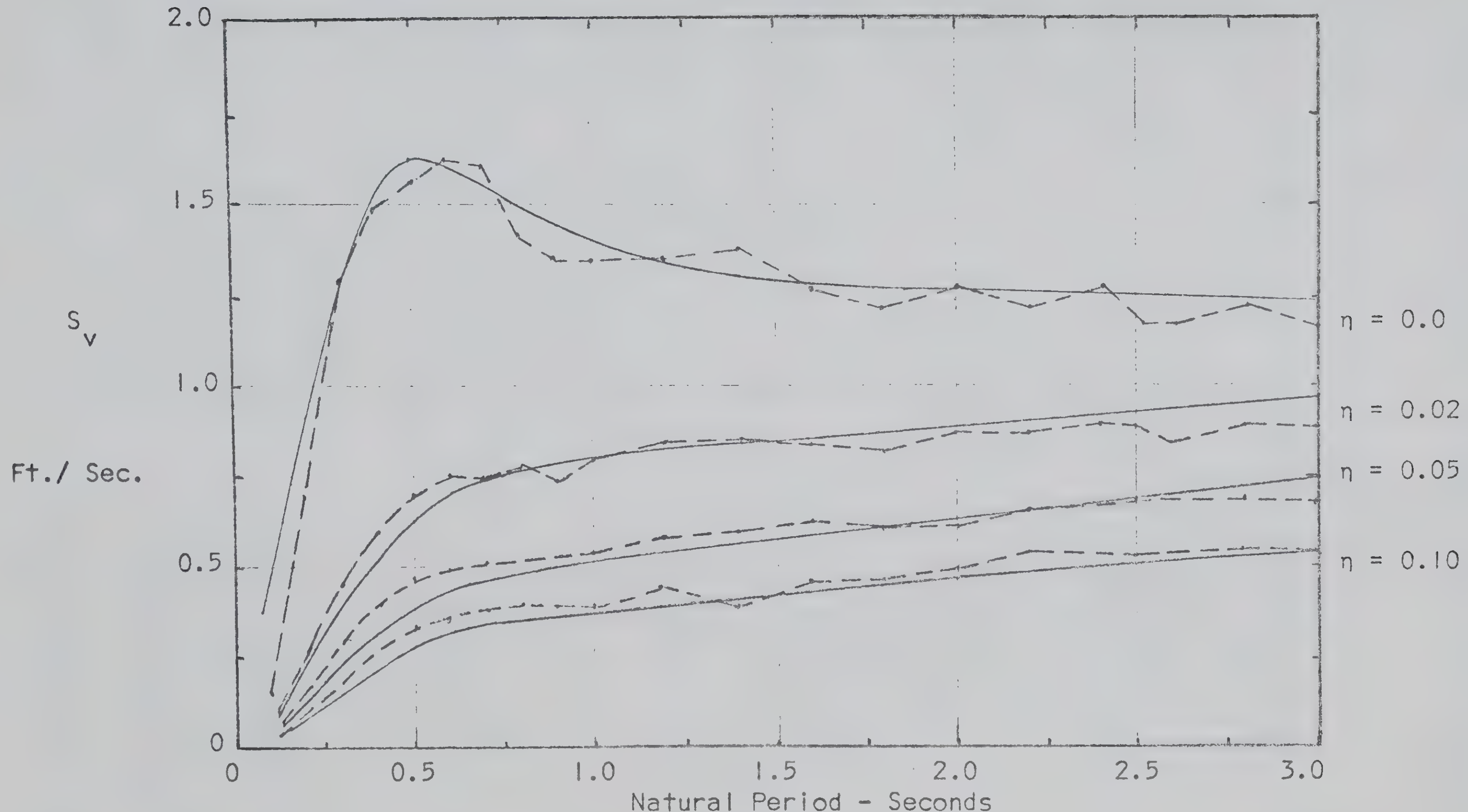

Figure 2.9 Average Response Spectra of Filtered White Noise Model Earthquakes Compared With the Spectrum Formula, Equation 2.5.

——— $1.796 \sqrt{\frac{\pi S(\omega)}{2\eta\omega}\left(1 - e^{-2\eta\omega\frac{30}{2.44}}\right)}$ - - - - - Artificial Earthquake Spectra

CHAPTER III

ROCKING BLOCK EXCITED BY RANDOM IMPULSE MODEL EARTHQUAKES

3.1 Free Vibration of a Rocking Block

Consider a rigid block as shown in Figure 3.1 of weight, W, mass, M, moment of inertia, I_o, about an axis through the corner 0, and with the angle α between the diagonal and the side of the block defining its proportions and referred to as the slenderness ratio. When such a block is given an angular displacement, θ, measured from the vertical as shown, then providing $\theta < \alpha$, the weight will cause a restoring moment about 0 of magnitude $WR \sin(\alpha - \theta)$ where R is the radial distance along the diagonal from 0 to the center of gravity. Taking clockwise as the positive direction for angular displacement, the equation of motion for positive displacement can be written as

$$I_o \ddot{\theta} = -WR \sin(\alpha - \theta), \quad 0 \leq \theta \leq \alpha$$

For tall, slender blocks such that the angle α is small enough to use the approximation $\sin(\alpha - \theta) \approx \alpha - \theta$, this equation becomes

$$I_o \ddot{\theta} - WR\theta = -WR\alpha \tag{3.1}$$

A property of the block, denoted q, which is analogous to the natural circular frequency of a harmonic oscillator may by defined such that $q^2 = \frac{WR}{I_o}$. For a uniform solid block $I_o = \frac{4}{3} MR^2 = \frac{4}{3} \frac{WR^2}{g}$ so that $q^2 = \frac{3g}{4R}$. Because the value

of q depends only on the length R it will be referred to as the size parameter of the block. Dividing Equation 3.1 by I_o yields $\ddot{\theta} - q^2\theta = -q^2\alpha$.

In terms of a non-dimensional quantity ϕ, defined as $\phi = \frac{\theta}{\alpha}$, the equation of motion becomes

$$\ddot{\phi} - q^2\phi = -q^2 \qquad 0 \leq \phi \leq 1 \qquad 3.2$$

For initial conditions $\phi = \phi_o$ and $\dot{\phi} = 0$ at $t = 0$, which correspond to the block being given an initial displacement $\theta_o = \phi_o\alpha$ and released, the solution of this equation may be shown to be

$$\phi = 1 - (1 - \phi_o) \text{ Cosh } qt, \quad 0 \leq \phi \leq 1 \qquad 3.3$$

The above derivation is applicable only for $\theta \geq 0$. Thus Equation 3.3 describes the block's motion only as it rotates back into the vertical position. The time t_1 at which the block would be in the vertical position can be determined by substituting $\phi = 0$ in Equation 3.3 to yield

$$t_1 = \frac{1}{q} \text{ Cosh}^{-1} \left(\frac{1}{1-\phi_o}\right) \qquad 3.4$$

If there is no energy loss during the impact at $\theta = 0$, the block will rotate about the corner at 0' until $\theta = -\theta_o$ and then fall back to the vertical position. To describe this phase of the block's motion, it is necessary to determine a different equation of motion and its solution subject to initial conditions of zero displacement and known initial angular velocity at $t = t_1$. The equation

of motion for negative displacement can be written as

$$I_o\ddot{\theta} = WR\sin(\alpha + \theta), \quad -\alpha \leq \theta \leq 0$$

With q and ϕ defined as above, this equation becomes

$$\ddot{\phi} - q^2\phi = q^2, \qquad -1 \leq \phi \leq 0 \qquad 3.5$$

For initial conditions $\phi = 0$ and $\dot{\phi} = \dot{\phi}_o$ at $t = t_1$, the solution of this equation is

$$\phi = -1 + \frac{\dot{\phi}_o}{q}\ \mathrm{Sinh}\ q(t-t_1) + \mathrm{Cosh}\ q(t-t_1), \qquad 3.6$$
$$-1 \leq \phi \leq 0$$

Equation 3.6 would describe the block's motion as it rotates from $\theta = -\theta_o$ and back to the vertical.

Each subsequent half-cycle of the block's motion could in turn be described by either Equation 3.2 or Equation 3.5 depending on whether the block was moving on the positive or the negative side of the vertical.

For the case of negligible energy loss, the block's motion would be symmetrical about the vertical position and the time t_1 as determined by Equation 3.4 would be equal to one quarter of the period of free vibration, T, so that

$$T = \frac{4}{q}\ \mathrm{Cosh}^{-1}\left(\frac{1}{1-\phi_o}\right)$$

Thus the time period of the vibration is dependent on the amplitude.

For the case when the energy loss at impact is not negligible, the angular velocity of the block after impact would be less than that just prior to impact and a velocity

reduction factor, $r \leq 1.0$, could be defined such that $r = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{\dot{\phi}_2}{\dot{\phi}_1}$ where $\dot{\theta}_1$ is the angular velocity before impact and $\dot{\theta}_2$ is the angular velocity after impact. Due to the loss of energy, the amplitude and hence the period of each half-cycle would be less than those of the preceding half-cycle. The free vibration of the block then would consist of initial oscillations of large amplitude and long periods with amplitude and period decreasing for each subsequent half-cycle until the block comes to rest in the vertical position.

3.2 Overturning By Random Impulses as Per Housner

As it was shown above, the motion of a rocking block is stepwise linear with the discontinuity between steps caused by the necessity to alternate between two different equations of motion. In considering random vibration problems, the statistical properties of the response process can usually be determined from those of the excitation process by means of a relationship determined from the differential equation describing the motion. For the case of a rocking block, there is no single differential equation of motion and such a relationship cannot be determined which would apply continuously so that it is not possible to apply the usual methods of statistical dynamics. Thus a statistical description of the block's motion cannot

be determined from the properties of the random impulse excitation.

Housner (1) in considering the problem of a rocking block subjected to random impulses was able to produce some measure of the probability of the block overturning by considering the average energy input due to the impulses in an analysis similar to the following.

Consider a block such as is shown in Figure 3.2 with the base on which it is standing being subjected to a random impulse model earthquake excitation, $\ddot{x}$, consisting of impulses, $v_i = \pm \Delta v$, occurring at random time intervals. The effect of a horizontal acceleration impulse, v_i, would be equivalent to an impulsive inertial force, $F_i \Delta t$, acting through the center of gravity of the block such that

$$F_i \Delta t = M v_i = \pm M \Delta v, \quad i = 1,2,3,\ldots..$$

where the subscript i denotes the effect of the ith impulse and M is the mass of the block. This force impulse causes a moment impulse about the corner 0 of magnitude $F_i \Delta t \; R \cos(\alpha - \theta)$. Limiting consideration to tall slender blocks for which α is small enough that $\cos(\alpha - \theta)$ may be taken equal to unity and from impulse momentum considerations of the rotational motion of the block about the axis 0 we obtain

$$I_o \, \Delta\theta_i = R F_i \Delta t$$

Thus the step change in the angular velocity of the block due to the ith ground impulse may be determined as:

$$\Delta\dot{\theta}_i = \frac{RF_i\Delta t}{I_o} = \frac{RMv_i}{I_o} \qquad 3.7$$

The kinetic energy of the block would undergo an increment change due to the ith impulse of

$$\Delta K_i = \frac{1}{2} I_o (\dot{\theta}_i + \Delta\dot{\theta}_i)^2 - \frac{1}{2} I_o(\dot{\theta}_i)^2$$

$$= I_o \cdot \Delta\dot{\theta}_i \cdot \dot{\theta}_i + \frac{1}{2} I_o(\Delta\dot{\theta}_i)^2$$

$$= MR \cdot v_i \cdot \dot{\theta}_i + \frac{1}{2} \frac{(MR)^2}{I_o}(v_i)^2 \qquad 3.8$$

The average change in energy per impulse is found by averaging over all impulses to obtain

$$\Delta K = MR \cdot \overline{v_i} \cdot \overline{\dot{\theta}_i} + \frac{1}{2} \frac{(MR)^2}{I_o} \cdot \overline{(v_i)^2}$$

But there is equal probability of any one impulse being positive or negative so that the average over all impulses would give $\overline{v_i} = 0$ and since for any impulse $(v_i)^2 = (\Delta v)^2$, then $\overline{(v_i)^2} = (\Delta v)^2$. Thus the average increment in energy becomes

$$\overline{\Delta K} = \frac{1}{2} \frac{(MR)^2}{I_o} \cdot (\Delta v)^2$$

If there were no energy loss during rocking, the energy of the block would increase so that on the average, the block's energy at the end of N impulses would be

$$E_N = \overline{\Delta K} \cdot N = \frac{1}{2} \frac{(MR)^2}{I_o} \cdot (\Delta v)^2 \cdot N \qquad 3.9$$

As the energy of the block increases so does its amplitude of vibration. The maximum amplitude the block can attain without overturn-

ing is α and to vibrate with this amplitude the energy of the block must be equal to the change in potential energy as the block moves from the vertical position to $\theta = \alpha$, that is

$$P = WR(1 - \cos \alpha) = \frac{1}{2} WR\, \alpha^2 \qquad 3.10$$

From Equations 3.9 and 3.10 it can be determined that on the average, the condition for the block to be vibrating with sufficient energy to attain an amplitude α after N impulses is

$$\frac{1}{2} WR\, \alpha^2 = \frac{1}{2} \frac{(MR)^2}{I_o} (\Delta v)^2 N$$

or

$$\Delta v \sqrt{N} = \alpha \sqrt{\frac{I_o}{MR^2} \cdot Rg} \qquad 3.11$$

When Equation 3.11 is satisfied, then the probability of the block overturning would be equal to the probability of the next impulse being in the direction to increase the block's energy. As there is equal probability of each impulse being in either direction, Equation 3.11 represents the condition for 50 per cent probability that the block will overturn.

In chapter two it was shown that for a random impulse model earthquake the theoretical undamped response spectrum would be

$$S_{vt}(0,\omega) = \Delta v \cdot \sqrt{N}$$

In terms of the response spectrum, Equation 3.11 may be written as

$$S_{vt}(0,\omega) = \alpha \sqrt{\frac{I_o}{MR^2} \cdot Rg} \qquad 3.12$$

For a particular block subjected to a ground motion

with a given response spectrum, it is only possible to determine from the above theory whether the probability of overturning is greater than or less than 50 per cent.

For a uniform solid block $I_o = \frac{4}{3} MR^2$ and Equation 3.12 becomes

$$S_{vt}(0,\omega) = \alpha\sqrt{\frac{4}{3} . Rg} \qquad 3.13$$

relating the size and proportions of a block to the response spectrum of the earthquake for which it would have a 50 per cent probability of overturning. This relationship shows that for constant α the larger the value of R the stronger the earthquake that may be survived. That is, for two blocks of the same proportions, the taller block would be more stable than the shorter. This can also be explained by substituting $I_o = \frac{4}{3} MR^2$ into Equation 3.7 to obtain

$$\Delta\dot{\theta}_i = \frac{RM}{I_o} . v_i = \frac{3}{4R} . v_i$$

which shows that the angular impulse $\Delta\dot{\theta}_i$ imparted to a block by a ground impulse v_i is inversely proportional to the size of the block.

3.3 Survival Probabilities Determined by Simulation

To test Housner's theoretical results and to develop more complete data on the probability of a block remaining upright when subjected to earthquake ground motion modeled as random impulses, the system shown in Figure 3.2 was simulated on the digital computer for various values of the slenderness

ratio, α, the block size parameter, q, and the earthquake magnitude parameter, S_V. Each individual simulation consisted of calculating the details of a given block's motion when subjected to a particular random impulse model earthquake of a given magnitude to determine whether or not the block would overturn. From the number of survivals out of 1000 such simulations, each with a different earthquake, the probability of the block not overturning, termed the survival rate, was determined for that particular block and earthquake magnitude. By varying the parameters, results were obtained for the significant range of block sizes and earthquake magnitudes.

In each simulation, the excitation, $\ddot{x}$, consisted of a different random impulse model earthquake with a duration of 30 seconds generated as outlined in Section 2.1.2. The undamped velocity response spectrum for these earthquakes was found to be $S_V(0,\omega) = \frac{8}{7} \cdot 10.\Delta v$ so that earthquakes of varying spectrum value were produced by varying Δv, the magnitude of the impulses. As shown in Equation 3.7 an acceleration impulse Δv causes a step change in the angular velocity of the block

$$\Delta\dot{\theta} = \frac{RM}{I_o} \cdot \Delta v = \frac{q^2}{g} \cdot \Delta v$$

where as previously defined $q^2 = \frac{WR}{I_o} = \frac{3g}{4R}$. In terms of the non-dimensional displacement function $\phi = \theta/\alpha$, the step change in velocity is

$$\Delta\dot{\phi} = \frac{\Delta\dot{\theta}}{\alpha} = \frac{q^2}{g} \cdot \frac{\Delta v}{\alpha} \qquad 3.14$$

Equation 3.14 shows that blocks which have the same parameter q (i.e. the same semi-diagonal R) but have different slenderness ratios α will experience the same magnitude of angular impulse $\Delta\dot{\phi}$ as long as they are subjected to model earthquakes with impulse magnitude Δv such that $\Delta v/\alpha$ is the same for all blocks. Since the block's motion in terms of ϕ depends directly on the magnitude of the impulse $\Delta\dot{\phi}$, it was to be expected that the survival rate of a block with given value of q would depend on the ratio $\Delta v/\alpha$ or since the response spectrum is defined in terms of Δv alone, on the ratio S_v/α. Thus by determining the survival rate for a particular ratio S_v/α and various values of the size parameter q, the coordinates were obtained to plot a curve of survival rate versus q. A number of such curves for different values of S_v/α were obtained by means of which the expected survival rate for any particular block and earthquake were shown.

In each simulation to determine the motion of a given block due to a particular earthquake, the total time of the motion was divided into a number of intervals within each of which the block's motion was determined as in initial condition problem. The effect of each ground impulse of magnitude Δv was to cause a step change in the angular velocity, or an angular impulse of magnitude $\Delta\dot{\phi}$. Between impulses the block moves as in free vibration so that its motion consists of a consecutive series of free vibration problems, each one

starting just after the occurrence of one impulse and ending just before the next impulse. The initial conditions in each case would be the final conditions of the previous interval plus the effect of the impulse. That is, the initial displacement would be the displacement at the time of the impulse and the initial velocity would be the velocity at the time of the impulse plus the step change in velocity due to the impulse.

In considering the free vibration of a rocking block, it was found that two equations of motion were required, one which held for positive displacements and one for negative displacements of the block. This meant that in addition to a new initial condition problem being initiated by each impulse, it was also necessary to consider a new problem each time the block passed through the vertical (zero displacement) position. The initial conditions for this case would be zero displacement and initial velocity equal to the velocity just prior to impact multiplied by the velocity reduction factor, r, to account for energy loss at impact. If energy losses were considered negligible r could be taken as unity.

The two equations of motion

$$\ddot{\phi} - q^2\phi = -q^2 \qquad \text{for} \quad 0 \leq \phi \leq 1.0$$

and

$$\ddot{\phi} - q^2\phi = q^2 \qquad \text{for} \quad -1 \leq \phi \leq 0$$

were replaced by one equation

$$\ddot{\phi} - q^2\phi = Hq^2$$

by defining H as $H = -1$ when $\theta \geq 0$

$$H = 1 \quad \text{when } \theta \leq 0$$

so that H would change value each time the block passed through the zero displacement position.

The initial condition problems were then defined as:

$$\ddot{\theta}_i(t) - q^2 \phi_i(t) = Hq^2 \quad \text{for } 0 \leq t \leq \Delta t_i \qquad 3.15$$

with $\phi_i(0) = \phi_{i-1}(\Delta t_{i-1})$

and $\dot{\phi}_i(0) = \dot{\phi}_{i-1}(\Delta t_{i-1}) + \Delta\dot{\phi}_i$

or $\dot{\phi}_i(0) = \dot{\phi}_{i-1}(\Delta t_{i-1}) \cdot r$

where the first velocity condition applied for an interval initiated by an impulse, in which $\Delta\dot{\phi}_i$ was the step change in velocity due to the impulse and the second condition for an interval initiated by passage through zero displacement, in which r is the velocity reduction factor. The subscript i denotes the interval and Δt_i is the length of the ith time interval.

In the generation of the model earthquake to be used in the simulation, the first ground impulse, v_i, was always taken as being positive, $v_1 = +\Delta v$, and subsequent impulses randomly selected as positive or negative, $v_i = \pm \Delta v$, as described in Section 2.1.2 Each ground impulse will cause a step change in velocity, $\Delta\dot{\phi}_i$, where

$$\Delta\dot{\phi}_i = \frac{q^2}{g} \cdot \frac{v_i}{\alpha}$$

The first ground impulse, v_1, will start the block moving with an

initial angular velocity $\Delta\dot{\phi}_1$. Setting v_1 positive ensured that in the first interval the block would experience positive displacement so that $H = -1$ and the initial conditions for the first interval were

$$\phi_1(0) = 0, \quad \dot{\phi}_1(0) = \Delta\dot{\phi}_1 = \frac{q^2}{g} \cdot \frac{\Delta v}{\alpha}$$

In each interval the solution for the displacement was taken in the form $\phi = 1 + Ae^{qt} + Be^{-qt}$ where the constants A and B were determined from the initial conditions. From the ratio A/B it was determined whether there was a possibility of a peak value, $\phi = \phi_{max}$, or a zero crossing, $\phi = 0$, before the next impulse. If no zero crossing occurred, then the interval would be terminated by the next impulse and the values of ϕ and $\dot{\phi}$ at the time of the impulse were calculated and used to determine initial conditions for the next interval.

As the block would overturn if the angle it makes with the vertical becomes greater in magnitude than α because the moment due to the weight would then be acting to increase the angle rather than restore the block to the vertical position, the necessary conditions for survival is that $|\phi|<1.0$ throughout the block's motion. Whenever the displacement was calculated, either at the end of an interval or as the peak value in an interval, a check was made to see if this condition was satisfied.

The calculations of the details of the block's motion

were continued until a value of displacement was determined such that $|\phi| \geq 1.0$ or until the last impulse of the model earthquake had been accounted for with either a failure or survival being recorded for that particular earthquake.

3.4 Results of Computer Simulation

In the curves of Figure 3.3 the survival rate is plotted against the block size parameter q for various values of the ratio S_v/α. These curves present the data as determined by computer simulation for the ideal case of perfectly elastic impact for which the velocity reduction factor, r, is equal to one.

For each particular value of S_v/α the curve extends from a minimum value of q such that for $q \leq q_{min}$ the survival rate will be 100% to a maximum value of q such that for $q \leq q_{max}$ the survival rate will be zero. Between these two extremes as q increases the survival rate decreases. For a uniform block it has been shown that $q^2 = \frac{3g}{4R}$ so that increasing q corresponds to decreasing R. A curve of R verses q is plotted in the top part of Figure 3.3 and superimposed on it are some sketches showing the sizes of blocks with $\alpha = 0.10$ corresponding to particular values of q. It is apparent that these curves show agreement with the general result predicted from consideration of Equation 3.13 derived by Housner's approximate theory that a taller block would be more stable than a shorter one of the same proportions. This can be dem-

onstrated by considering two different blocks, one 100 feet high by 10 feet wide and the other 25 feet by 2.5 feet, each subjected to an earthquake with $S_V = 3.5$ ft./sec. The slenderness ratio in each case is $\alpha = 0.10$ so the particular survival curve of interest is $S_V/\alpha = 35$. As shown by the dashed lines in Figure 3.3 the probability of the taller block surviving would be 79 % compared to a probability of only 11% for the shorter block.

Equation 3.13 predicts theoretically the response spectrum value for which a block would have a 50% survival rate. In terms of the size parameter, $q = \sqrt{\frac{3g}{4R}}$, this equation becomes

$$\frac{S_V}{\alpha} = \frac{g}{q} \qquad 3.16$$

The graph of Equation 3.16 is shown in Figure 3.4 where S_V/α is plotted against q. Corresponding to the various values of S_V/α in Figure 3.3, the value of q was determined for which the survival rate was 50% as determined by the computer simulations. These pairs of values were plotted as additional points in Figure 3.4 to show the agreement between the theory and the simulation results for this particular case of 50% survival probability.

Thus the limited results predicted by the approximate energy approach for the case of a random impulse excitation were confirmed by the data obtained from the computer simulation of the same situation. In addition, the simulation analysis gives results for a much broader range of circum-

stances.

Each survival rate value used in the data of Figure 3.3 was determined by subjecting a particular block to each of a randomly generated set of one thousand model earthquakes and determining how many earthquakes out of the sample set it would survive without overturning. If a different sample of one thousand earthquakes had been used, it would be expected that a different survival rate would result due to the statistical nature of the problem and the limitations imposed by a finite sample size.

To obtain an indication of the statistical consistency or "goodness" of the simulation results, one particular case, $S_v/\alpha = 25$, was repeated seven times, each time with a different set of model earthquakes. The variation in the survival rates obtained is shown in Figure 3.5 where the range of values obtained for certain block sizes is shown superimposed on the original curve for this case, $S_v/\alpha = 25$, from figure 3.3. From being negligible near the extremes of zero and 100% survival rate, the variation increases to a maximum in the area of 50% survival rate. For $q = 1.30$ the survival rates obtained ranged from 44.3% to 52.1% with a mean value of 47.7%. There was good general agreement among the results obtained from the eight different sets of earthquakes with no outstanding inconsistencies. A larger sample size would yield more reliable survival rates but to use

more than one thousand earthquakes would cause the required computer time to be prohibitive.

To illustrate the effect of energy loss due to impact, the survival data for $S_v/\alpha = 25$ was determined with the velocity reduction factor having the values $r = 0.95$ and $r = 0.90$. The curves thus obtained are also plotted in Figure 3.5 for comparison with the curve for the ideal case, $r = 1.0$, from Figure 3.3 to demonstrate that the effect of any energy loss is to increase the probability of a block's survival.

It would be difficult to assess what value r might have for any actual situation as it would depend on the material properties of the ground and the structure. For the case of completely inelastic impact application of the conservation of angular momentum yields an expression for r in terms of α, the slenderness ratio. Consider Figure 3.6 which shows a block just prior to impact rotating about the corner 0 with angular velocity $\dot{\theta}_1$ and angular momentum about 0' given by $(I_o - MR^2)\dot{\theta}_1 + MR\dot{\theta}_1 \cdot (h \cos \alpha - w \sin \alpha)$. After impact the block will continue its rotation about the corner 0' with velocity $\dot{\theta}_2$ and the angular momentum $I_{o'} \cdot \dot{\theta}_2$. Equating angular momentum about 0' just prior to impact to that just after impact yields

$$I_o\dot{\theta}_1 - MR\dot{\theta}_1[R - (h \cos \alpha - w \sin \alpha)] = I_{o'}\dot{\theta}_2 \qquad 3.17$$

But $R-(h \cos \alpha - w \sin \alpha) = 2w \sin \alpha = 2R \sin^2\alpha$ and $I_{o'} = I_o$

from the geometry of the block so Equation 3.17 becomes

$$I_o\dot{\theta}_1 - 2MR^2 \sin^2\alpha\dot{\theta}_1 = I_o\dot{\theta}_2 \qquad 3.18$$

For slender blocks such that sin $\alpha \approx \alpha$ the velocity reduction factor is obtained from Equation 3.18 to be

$$r = \dot{\theta}_2/\dot{\theta}_2 = 1 - 2MR^2\alpha^2/I_o$$

Substituting $I_o = \frac{4}{3} MR^2$ for a uniform block yields the relation

$$r = 1 - \frac{3}{2}\alpha^2$$

showing that the energy loss varies directly with the square of the slenderness ratio.

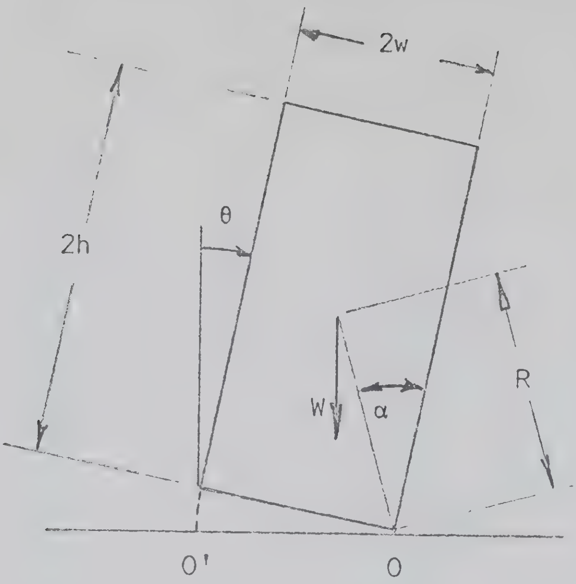

Figure 3.1 Rigid Block in Free Vibration.

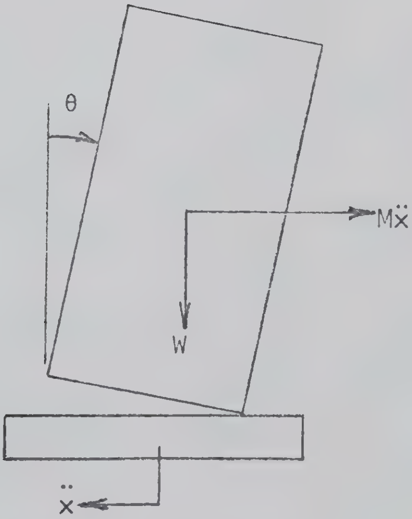

Figure 3.2 Rigid Block Subjected to Base Acceleration.

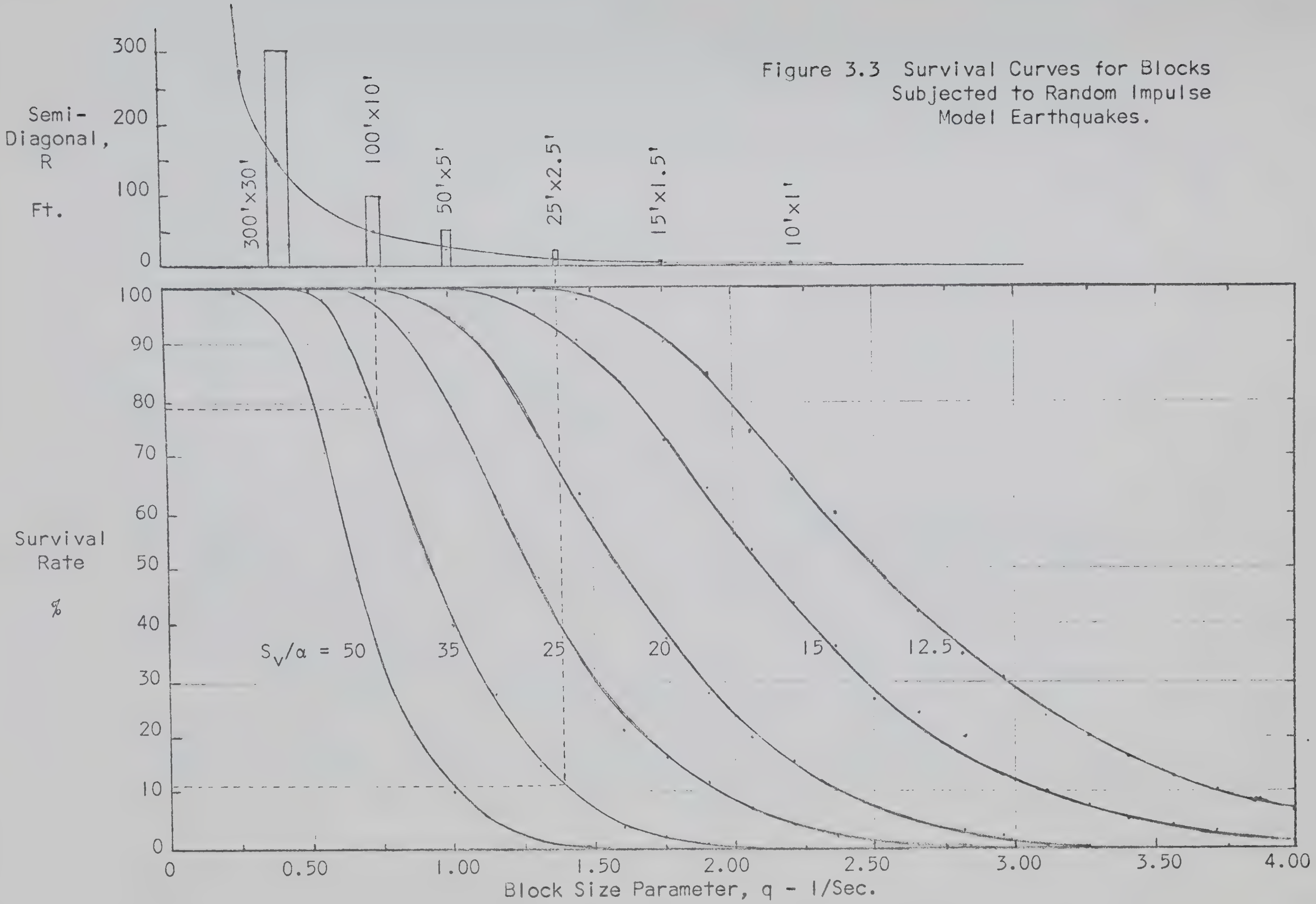

Figure 3.3 Survival Curves for Blocks Subjected to Random Impulse Model Earthquakes.

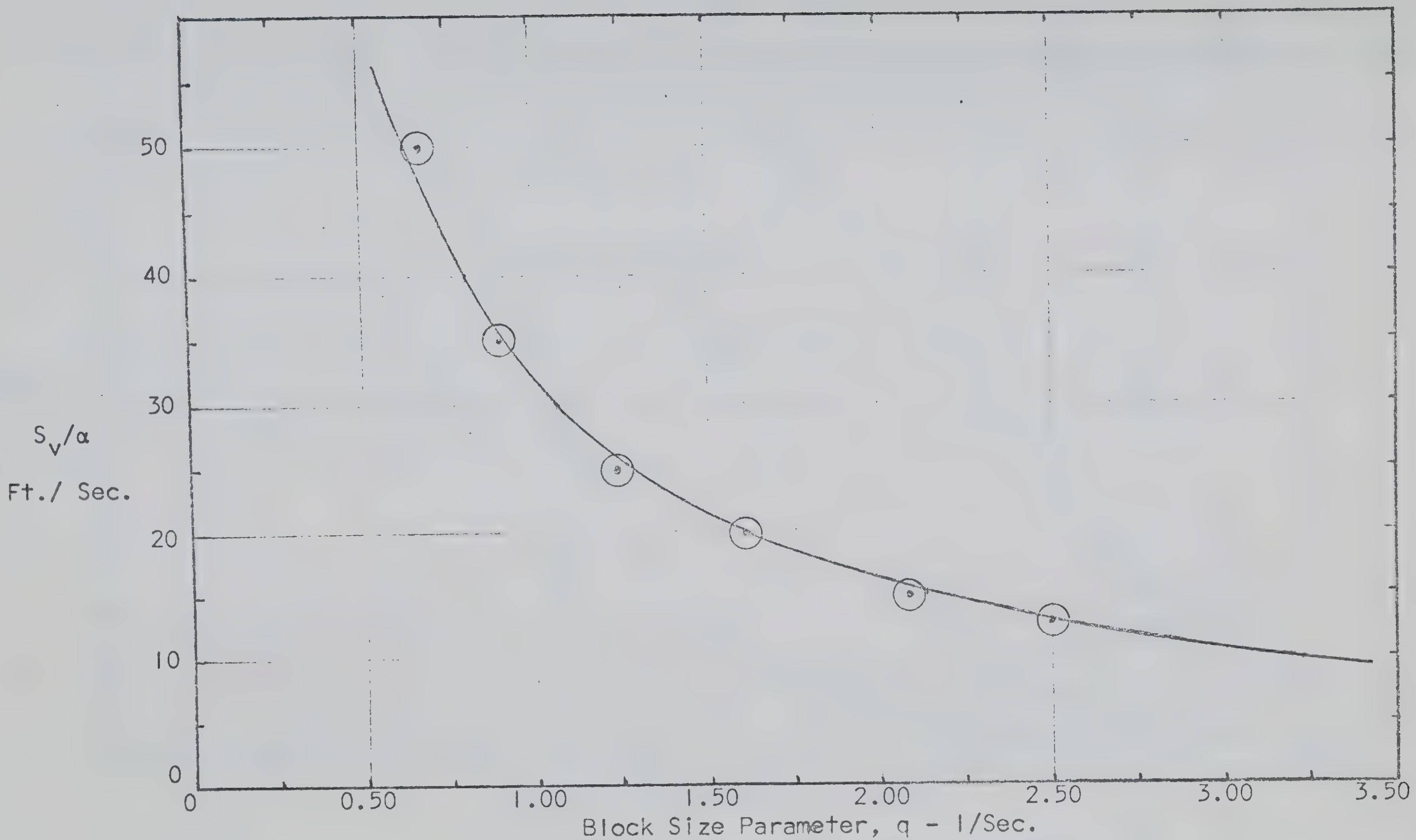

Figure 3.4 Computer Simulation Results Compared to Theoretical Prediction for 50% Survival

—— Graph of Equation 3.16

⊙ Data from Computer Simulation

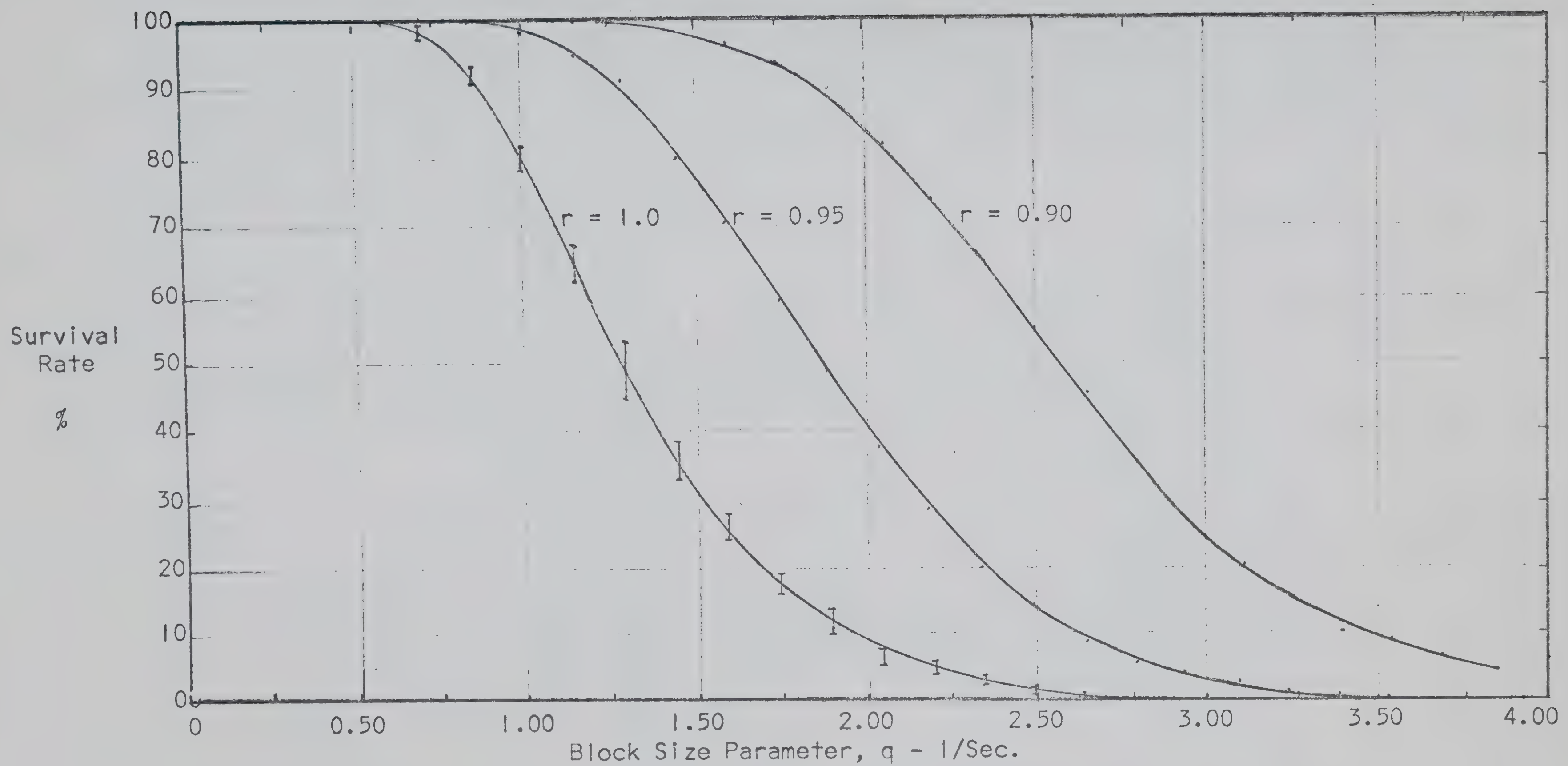

Figure 3.5 1. Variation in Survival Rates for Repeated Simulations. (indicated by bar graphs on r = 1.0 curve)

2. The Effect of Energy Loss At Impact.

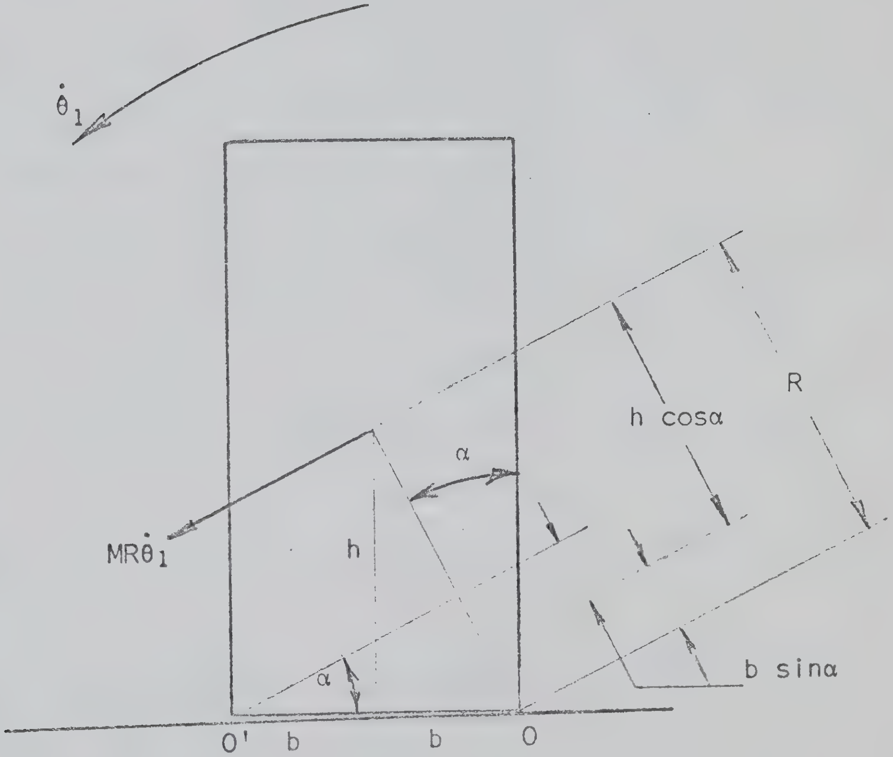

Figure 3.6 Block Just Prior to Impact.

CHAPTER IV

ROCKING BLOCK EXCITED BY FILTERED WHITE NOISE MODEL EARTHQUAKE

4.1 Introduction

To obtain a different appraisal of the probability of the rocking block surviving without overturning in an earthquake, the system of Figure 3.2 was simulated on the computer using filtered white noise model earthquakes for the ground acceleration $\ddot{x}$.

The ensemble of fifty standard earthquake accelerograms produced as outlined in chapter two was used for the excitation. Each earthquake was defined by 1200 values of the acceleration at intervals of .025 seconds, connected by straight lines to produce a segmentally linear function. To vary the severity of the earthquake ensemble, the acceleration values were multiplied by a scale factor Q.

For a given strength of earthquake, i.e. a given value of Q, a block with a particular slenderness ratio α and size parameter q was subjected to each of the fifty earthquakes in turn. For each earthquake the motion of the block was calculated to determine whether the block would overturn or not. From the number of survivals out of the fifty earthquakes, the survival rate was determined for that particular block and earthquake magnitude.

4.2 Forced Vibration of a Rocking Block.

Consider a rigid block with properties α, R and W as

defined in chapter three and shown in Figure 3.1, free to rock on the ground which is experiencing acceleration $\ddot{x}(t)$ as shown in Figure 3.2. The effect of the ground acceleration can be accounted for by an inertia force, $\frac{W}{g} \cdot \ddot{x}$, acting through the center of mass of the block as shown. Taking clockwise as the positive direction for angular displacement, θ, and $\ddot{x}$ positive in the direction shown, the equation of motion for positive displacement is

$$I_o\ddot{\theta} = -WR\sin(\alpha - \theta) + \frac{W}{g}\ddot{x}R\cos(\alpha - \theta), \quad 0 \leq \theta \leq \alpha \qquad 4.1$$

For tall slender blocks, the approximations $\sin(\alpha - \theta) \simeq \alpha - \theta$ and $\cos(\alpha - \theta) \simeq 1$ may be used and with $q^2 = \frac{WR}{I_o}$ as defined in chapter three, Equation 4.1 becomes

$$\ddot{\theta} - q^2\theta = -q^2\alpha + q^2 \cdot \frac{\ddot{x}}{g}$$

In terms of the non-dimensional quantity ϕ defined as $\phi = \frac{\theta}{\alpha}$ the equation of motion for positive displacements becomes

$$\ddot{\phi} - q^2\phi = -q^2 + q^2 \cdot \frac{\ddot{x}}{g\alpha}, \quad 0 \leq \phi \leq 1 \qquad 4.2$$

Because of the change in the direction of the restoring moment due to the weight, a different equation describes the motion when the block is experiencing negative displacement. This equation is:

$$I_o\ddot{\theta} = WR\sin(\alpha + \theta) + \frac{W}{g}\ddot{x}R\cos(\alpha + \theta) \qquad -\alpha \leq \theta \leq 0$$

In terms of q and ϕ this equation becomes

$$\ddot{\phi} - q^2\phi = q^2 + q^2\frac{\ddot{x}}{g\alpha}, \quad -1 \leq \phi \leq 0 \qquad 4.3$$

If the block, initially at rest, is subjected to a positive base acceleration, $\ddot{x} > 0$, it would tend to experience a positive displacement and Equation 4.2 would apply. When $\phi = 0$, it can be seen from Equation 4.2 that a necessary condition to obtain a positive angular acceleration ($\ddot{\phi} > o$) and hence start the block rotating is $\ddot{x} > g\alpha$. That is, the base acceleration must be greater than the product of the gravitational acceleration and the slenderness ratio. If the ground acceleration is such that $0 < \ddot{x} < g\alpha$ the block will experience a translational motion with the ground but it will not rotate.

Similarly for the case of negative ground acceleration, $\ddot{x} < 0$, consideration of Equation 4.3 with $\phi = 0$ reveals that for the block to acquire a negative angular acceleration, $\ddot{\phi} < 0$, it is required that $\ddot{x} < -g\alpha$.

Thus the condition on the magnitude of ground acceleration required to initiate rotational motion of the block can be stated as $|\ddot{x}| > g\alpha$.

By using the variable H as defined in chapter three, the two Equations 4.2 and 4.3 were replaced by one equation

$$\ddot{\phi} - q^2\phi = Hq^2 + q^2 \frac{\ddot{x}}{g\alpha} \quad , \quad -1 \leq \phi \leq 1 \qquad 4.4$$

with $H = -1$ when $\phi \geq 0$ and $H = +1$ when $\phi < 0$.

The solution to Equation 4.4 for the nondimensional displacement function $\phi(t)$ depends on the type of function defining $\ddot{x}(t)$, the ground acceleration.

The particular case of interest is ground acceleration defined as a linear function,

$$\ddot{x}(t) = a + bt \qquad 4.5$$

Substitution into Equation 4.4 yields

$$\ddot{\phi} - q^2\phi = Hq^2 + q^2\left(\frac{a+bt}{g\alpha}\right) \qquad 4.6$$

Defining $C = H + \frac{a}{g\alpha}$ and $D = \frac{b}{qg\alpha}$ yields

$$\ddot{\phi}\; q^2\phi = q^2(C + D\cdot qt) \qquad 4.7$$

for which the general solution can be taken to be

$$\phi(t) = Ae^{qt} + Be^{-qt} - C - D\cdot qt \qquad 4.8$$

The coefficients A and B may be determined by the initial values of the displacement, $\phi(0)$, and velocity, $\dot{\phi}(0)$.

4.3 Computer Simulation of a Rocking Block Subjected to Filtered White Noise.

The time record of the filtered white noise ground acceleration consists of consecutive time intervals of length $\Delta t = 0.025$ seconds within each of which the acceleration is a linear function. Within the ith time interval, $(i = 1,2,..1200)$ which extends from $t_i = (i - 1)\Delta t$ to $t_{i+1} = i\Delta t$ the acceleration was defined by values R_i and R_{i+1} at times t_i and t_{i+1} respectively and expressed as

$$\ddot{x}_i(t) = R_i + \frac{(R_{i+1}-R_i)}{\Delta t} \cdot t \qquad 4.9$$

with t as the time measured from the beginning of the interval so that $0 \leq t \leq \Delta t$.

The motion of the block was calculated as a series of consecutive initial condition problems with the displacement and velocity at the end of one interval serving as the initial conditions for the next interval. Using Equation 4.7 with C and D defined from comparison of Equations 4.9 and 4.5 as $C = H + \frac{R_i}{g\alpha}$ and $D = \frac{(R_{i+1}-R_i)}{\Delta t q g\alpha}$, the initial condition problems were defined as:

$$\ddot{\phi}_i(t) - q^2\phi_i(t) = q^2(C + Dqt), \quad \text{for } 0 \leq t \leq \Delta t \qquad 4.10$$

with
$$\phi_i(0) = \phi_{i-1}(\Delta t)$$

and
$$\dot{\phi}_i(0) = \dot{\phi}_{i-1}(\Delta t)$$

As was shown in Section 4.2 no rocking of the block will occur until the magnitude of the ground acceleration becomes large enough. If the initial value of the acceleration satisfies the condition $|R_1|>g\alpha$, the proper value was assigned to H ($H = -1$ if $R_1>0$, $H = 1$ if $R_1<0$) and the motion in the first interval would be determined from Equation 4.10 with starting conditions $\phi(0) = \dot{\phi}(0) = 0$.

If $|R_1|<g\alpha$, then it became necessary to determine the first interval within which $|\ddot{x}|$ becomes equal to $g\alpha$. If R_{j+1} was the first value of R_i for which $|R_{j+1}|>g\alpha$, then motion of

the block would begin in the jth interval at the time t_o in that interval when $|\ddot{x}_j(t_o|=g\alpha$. Measuring time, t, from the start of motion, then within the subinterval of length, $\Delta t' = \Delta t - t_o$ from the start of motion to the end of the jth interval, the acceleration would be $\ddot{x}_j(t) = R_o + \frac{(R_{j+1}-R_o)}{\Delta t'} \cdot t$

in which R_o is the value of ground acceleration when motion starts. If the ground acceleration is positive, $R_o = g\alpha$ and $H = -1$ and the constant C in Equation 4.7 would be

$$C = H + \frac{R_o}{g\alpha} = -1 + \frac{g\alpha}{g\alpha} = 0$$

If the ground acceleration is negative, $R_o = -g\alpha$ and $H = 1$ and again $C = 0$. Defining D as $\frac{(R_{j+1}-R_o)}{\Delta t' q g\alpha}$ the motion in the first sub-interval was determined from

$$\ddot{\phi}_j(t) - q^2\phi_j(t) = q^2Dqt \ , \qquad 0 \leq t \leq \Delta t'$$

with $$\phi_j(0) = \dot{\phi}_j(0) = 0 \ .$$

In each interval or subinterval, the solution for the displacement was taken in the form

$$\phi(t) = Ae^{qt} + Be^{-qt} - C - Dqt \qquad 4.11$$

with the constants C and D as defined in terms of the excitation and the constants A and B determined from the initial conditions. Differentiation yields the corresponding expression for angular velocity:

$$\dot{\phi}(t) = q(Ae^{qt} - Be^{-qt} - D) \qquad 4.12$$

If the displacement attained a value such that $|\phi| \geq 1.0$ overturning of the block would occur. To check for failure it was necessary to know the magnitude of the displacement at any maximum or minimum peaks that occurred in an interval.

Peak values of ϕ will occur at times t_1 when $\dot{\phi}(t_1) = 0$. Reference to Equation 4.12 shows that t_1 can be determined from $0 = Ae^{qt_1} - Be^{qt_1} - D$. Multiplying through by e^{qt_1} and defining a new variable $S = e^{qt_1}$ yields the quadratic equation $0 = As^2 - Ds - B$ the roots of which are given by $S = \frac{1}{2A} (D \pm \sqrt{D^2 + 4AB}.)$ If $(D^2 + 4AB)<0$ no peaks occur in the interval. If $(D^2 + 4AB)>0$ the displacement function given by Equation 4.11 has two peaks.

If it was determined that one or both of the peaks occurred at times $t_1<\Delta t$, then the peak value(s) of ϕ were calculated to determine if overturning would occurr in the interval. If there were no peaks in the interval, the value of ϕ at the end of the interval, $\phi(\Delta t)$, was calculated.

Whenever the value of ϕ was calculated at a peak or at the end of an interval, it was necessary to check that the algebraic sign of ϕ had not changed compared to the previous value of ϕ calculated. A sign change would indicate that a zero value of ϕ had occurred in the interval. Whenever ϕ passes through zero (which corresponds to the block passing through the vertical position) the sign of the parameter H changes so that the expression for $\phi(t)$, Equation 4.11, is only valid up to the time

t_2 at which $\phi(t_2) = 0$.

To solve for the time t_2 when $\phi(t_2) = 0$, an iterative process, the Newton-Raphson method, was used to determine the root of the equation,

$$0 = Ae^{qt_2} + Be^{-qt_2} - C - Dqt_2$$

If ϕ_1 were the last value of the displacement calculated before the zero and ϕ_2 the first value after the zero so that ϕ_1 and ϕ_2 were opposite in algebraic sign, a first approximation to t_2 was made by determining the time t_{21} at which a linear function from ϕ_1 to ϕ_2 would be zero. Subsequent approximation to t_2 were calculated by the sequence

$$t_{2(n+1)} = t_{2n} - \frac{\phi(t_{2n})}{\dot{\phi}(t_{2n})}$$

with the iteration being continued until $|\phi(t_2)| < 10^{-5}$ then this value of t_2 was taken as the time at which the zero occurred.

Following a zero crossing at time t_2 in the jth time interval, a new initial condition problem was defined for motion in the sub-interval from t_2 to the end of the jth interval. The value of H would be changed to the negative of its previous value. Measuring time, t, from the zero crossing, then within the sub-interval of length $\Delta t' = \Delta t - t_2$, the acceleration would be

$$\ddot{x}_j(t) = R_o + \frac{(R_{j+1} - R_o)}{\Delta t'} \cdot t$$

in which R_o is the value of ground acceleration when $\phi = 0$.

The constants C and D were defined as

$$C = H + \frac{R_o}{g\alpha}$$

and

$$D = \frac{(R_{j+1} - R_o)}{\Delta t' q g \alpha}$$

and the motion in the sub-interval was determined from

$$\ddot{\phi}_j(t) - q^2\phi_j(t) = q^2(C + D \cdot qt), \text{ for } 0 \leq t \leq \Delta t'$$

with

$$\phi_j(0) = 0$$

and

$$\dot{\phi}_j(0) = \dot{\phi}(t_2) \cdot r$$

in which $\dot{\phi}(t_2)$ is the velocity just prior to impact and r is the velocity reduction factor as defined in chapter three.

Theoretically, as stated above, no matter how small the angular velocity of the block as it falls to the vertical (zero displacement) position, it would continue to rotate after impact. This leads to the possibility of the block rocking back and forth with very small amplitude and small periods of vibration, such that for practical purposes it could be considered to be standing still. To avoid calculating the details of such motion, which would involve extensive computer time due to the large numbers of peaks and zero crossings that would occur, a limiting value was placed on the velocity before impact and if the velocity was less than or equal to this limit, it was assumed that the block came to rest. This limiting velocity was taken to be 0.0001 so that if $|\dot{\phi}| \leq 0.0001$ the block was considered to stop. To initiate motion

of the block again would required the ground acceleration to be such that $|\ddot{x}|>g\alpha$ so that the process outlined previously would be repeated for values of R_i in subsequent intervals.

The calculations of the details of the block's motion were continued until a value of displacement was determined such that $|\phi|\geq 1$ or until the last interval of the pseudo-earthquake acceleration had been accounted for with either a failure or survival being recorded for that particular earthquake.

4.4 Results of Computer Simulation

In Figure 4.1 the survival data are presented for blocks subjected to the filtered white noise type of model earthquakes considering perfectly elastic impact. For various values of the ratio Q/α, where Q is the excitation magnitude parameter and α is the slenderness ratio, curves of survival rate versus size parameter q, are plotted.

From comparison with Figure 3.3 it is seen that there is good qualitative agreement between the data obtained using the two different models of earthquake ground acceleration.

To make a meaningful quantitative comparison it is necessary to determine a criterion which relates the theoretical response spectrum value, S_{vt}, which defines the magnitude of the random impulse excitation, to the scale factor Q used to define the magnitude of the filtered white noise excitation.

Comparison could be made on the basis of the undamped spectrum intensity SI_o, the area under the zero damped spectrum curve. As shown in Section 3.1, the natural frequency of a rocking block depends on its amplitude so during its motion a block will have a varying natural frequency. Since SI_o provides a measure of the earthquake's effect over a range of frequencies, it seems reasonable to use it as a basis for comparison. For the random impulse model earthquakes, the spectrum intensity is determined from the spectrum value as

$$SI_o = S_{vt} \cdot \frac{8}{7} \cdot 2.40 = 2.74\ S_{vt}$$

For the filtered white noise model earthquakes, the undamped spectrum intensity for the standard ensemble is 3.13 so for various values of the scale factor Q, the spectrum intensity is

$$SI_o = 3.13 \cdot Q$$

The relationship between Q and S_{vt} such that both types of earthquake models would have the same spectrum intensity is

$$3.13Q = 2.74\ S_{vt}$$

from which

$$Q = 0.875\ S_{vt} \qquad 4.13$$

Another possible method of comparison could involve the response spectra. Before a block will overturn it must be rocking with large amplitude and corresponding large natural period. For large natural periods the zero damped spectrum curve for the standard ensemble of filtered white noise model earthquakes levels off at a value of 1.25 so that for a given value of Q

$S_v = 1.25 \cdot Q$. For the random impulse earthquake, the spectrum has the same value at all periods, $S_v = \frac{8}{7} S_{vt}$. Equating these values so that both earthquake models have the same spectrum value corresponding to long time period yields

$$1.25\ Q = \frac{8}{7} S_{vt}$$

or

$$Q = 0.915\ S_{vt}$$

It is seen that these two methods of camparison yield criteria which agree within five per cent. Using Equation 4.13, corresponding values of the ratios S_{vt}/α and Q/α to be used for comparisons were determined and are shown in Table 4.1. The second row shows the value of Q/α that should be used according to Equation 4.12 and the third row shows the nearest value of Q/α available from the data.

TABLE 4.1

CORRESPONDING VALUES OF S_{vt}/α and Q/α

S_{vt}/α	16.7	20	35
$.875 \cdot S_{vt}/\alpha$	14.6	17.5	30.6
Q/α	15	17.5	30

To show the comparison of the corresponding cases of

Table 4.1, three pairs of curves are shown in Figure 4.2. In most cases, the results obtained using the random impulse model earthquakes are more conservative than those using the filtered white noise model. That is, for a given value of the size parameter, q, the probability of survival as read from the S_{vt}/α curve is less than that given by the Q/α curve. As the ratios Q/α and S_{vt}/α decrease, this difference between corresponding cases becomes greater.

One factor that could cause this difference is the possibility of a block not being set into rocking motion by the finite accelerations of the filtered white noise. As discussed in Section 4.2, the block will not rock until the ground acceleration, $\ddot{x}(t)$, attains a value such that $|\ddot{x}|>g\alpha$. The ground acceleration magnitude depends directly on the scale factor, Q, so that decreasing Q enhances the possibility of the block not rocking for at least part of the earthquake's duration. In the case of the random impulse excitation, the acceleration attains infinite magnitude during each impulse so that the block must be rocking during the entire earthquake. Due to this major difference, it is to be expected that the impulses are a more severe form of excitation.

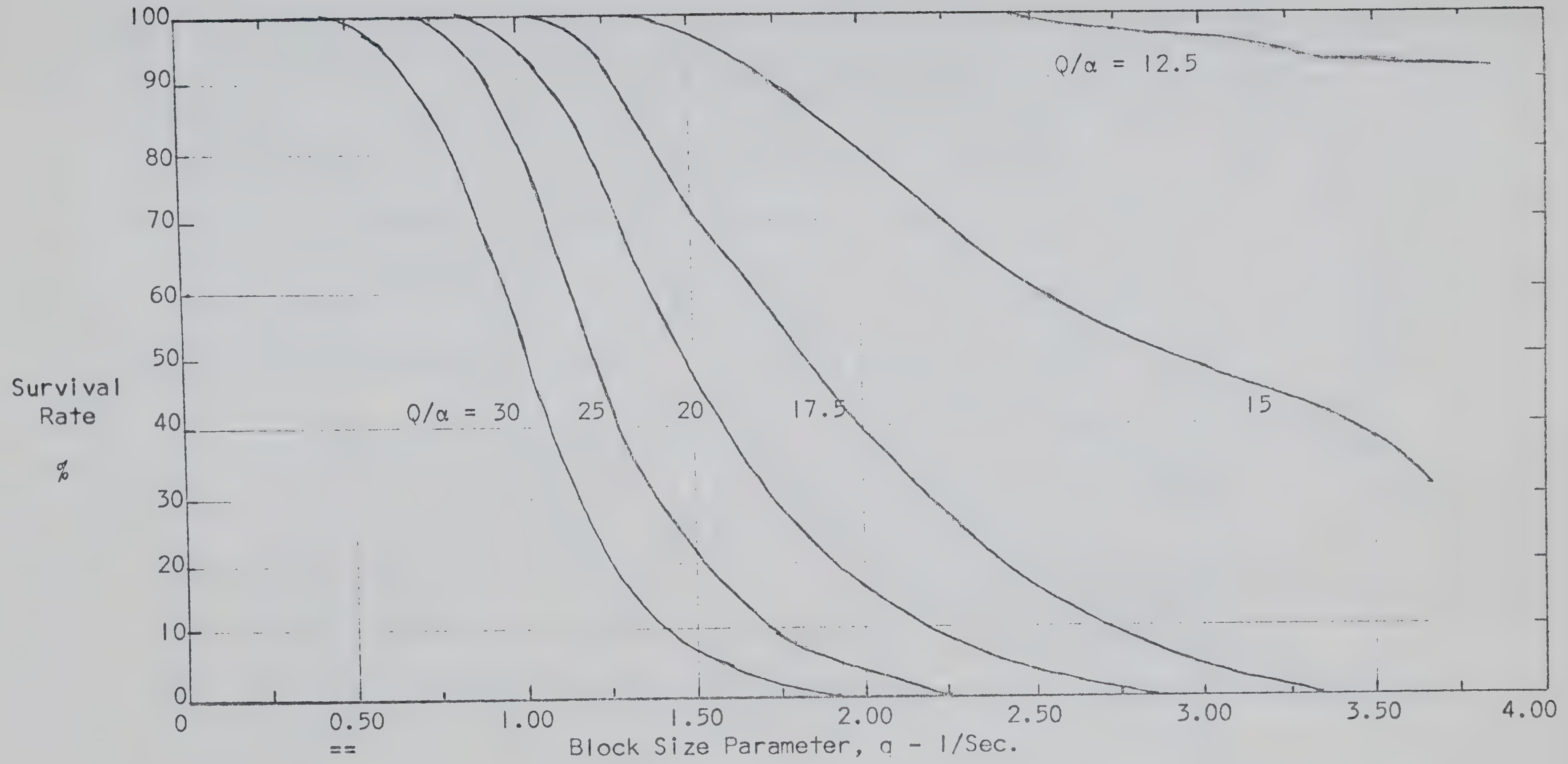

Figure 4.1 Survival Curves for Blocks Subjected to Filtered White Noise Model Earthquakes.

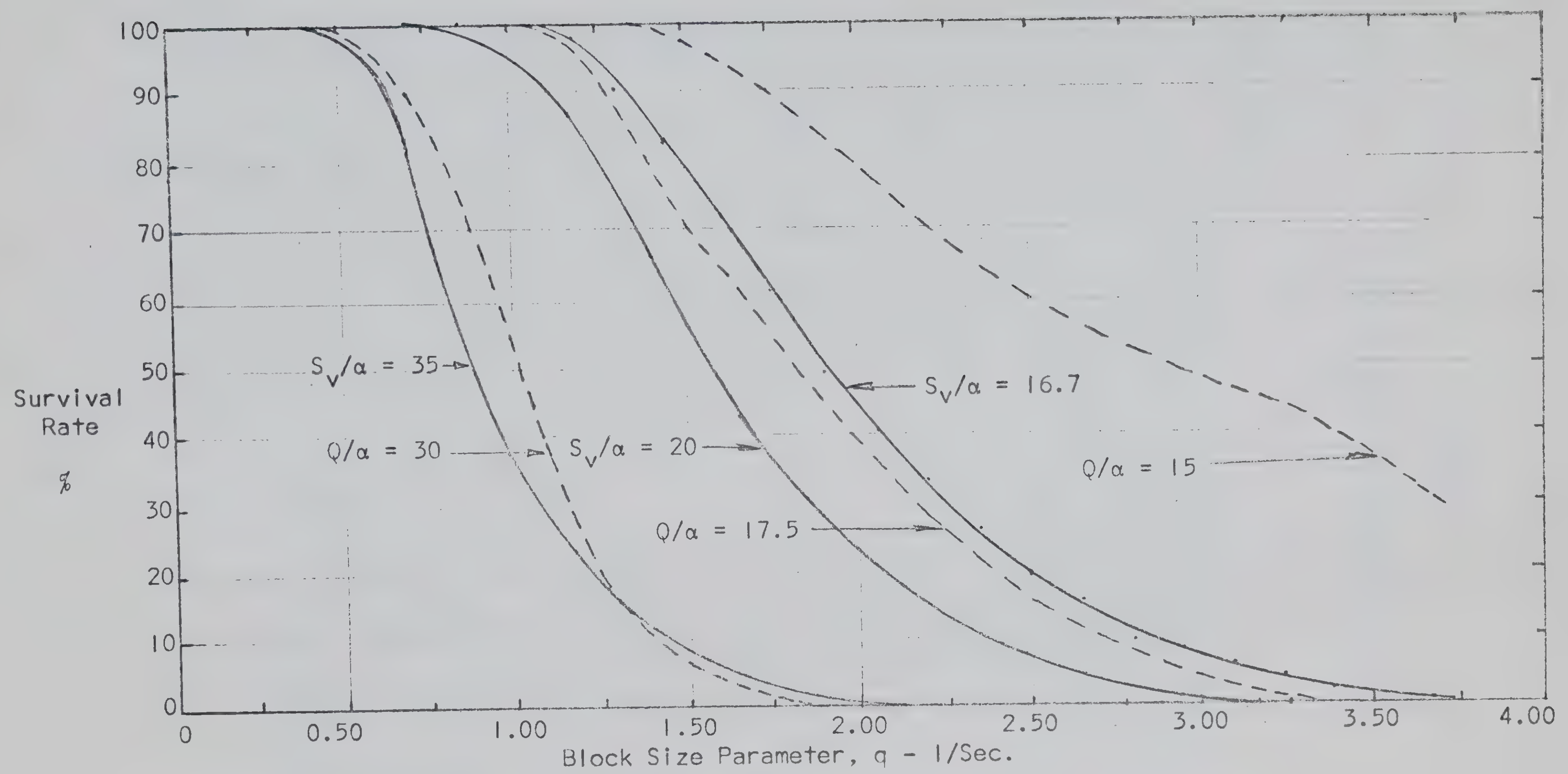

Figure 4.2 Comparison of survival Data for the Two Different Earthquake Models.

——— Random Impulse Type

– – – – – Filtered White noise Type

CHAPTER V

CONCLUSIONS and RECOMMENDATIONS

5.1 Summary

Using computer simulation, which consisted of calculating the details of the motion of the system on the digital computer, the probability of a free standing rigid block surviving an earthquake type disturbance without overturning was determined. Two different types of pseudo-earthquakes, (a) random impulses and (b) filtered white noise, produced as detailed in chapter two, were used for the ground acceleration. The data obtained for the case of random impulses was presented graphically in Figure 3.3 and found to be in agreement with results predicted by Housner's analysis (1). The data obtained for the case of filtered white noise was presented in Figure 4.1. Comparison of Figures 3.3 and 4.1 showed good agreement between the two types of excitation with regard to the general relationships between survival rate, block size and geometry, and excitation magnitude. With regard to the actual survival rates, it was found that when comparison was made on the basis of equal spectrum intensity that the random impulse model earthquake yielded more conservative values for the probability of survival.

5.2 Conclusions

1. Computer simulation provides a useful tool for determining response statistics for random processes, particularly for situations in which the statistical quantities required can not be calculated.

2. Data obtained by simulation confirmed two of Housner's results (1) for the case of a block subjected to random impulses: (a) the relationship between S_v, α and q for 50% probability of surviving (Equation 3.16), and (b) that if two blocks of the same proportion experience the same ground impulses the larger block will have a higher probability of surviving.

3. The probability of survival of a rocking structure can safely and conveniently be predicted by subjecting it to random impulse ground acceleration with a spectrum intensity equal to that of the expected earthquake. The impulse model earthquake is recommended for the following reasons:

(a) Although the filtered white noise type of model earthquake appears to be more realistic, it requires considerably more computer time, both to produce the model earthquake accelerogram and to calculate the blocks motion when subjected to it.

(b) The ease with which a random impulse model earthquake can be produced and the relative simplicity of calculating the resulting motion of the block makes it

possible to consider a larger sample and thus obtain more reliable statistical results.

(c) For the same spectrum intensity value, the random impulse model earthquake yields more conservative predictions for the survival probability.

5.3 Recommendations

1. Most of the survival data was obtained for the ideal case of no energy loss at impact between the block and the ground. It would be of interest to investigate the type of energy loss which may actually occur and its effect on survival.

2. The model earthquakes used in this paper were of 30 seconds duration. As the majority of earthquakes will cause strong ground motion for less than 30 seconds, the effect on the survival rate of varying the duration should be investigated.

BIBLIOGRAPHY

1. Housner, G.W., "The Behaviour of Inverted Pendulum Structures During Earthquakes", Bulletin, Seismological Society of America, Vol. 53, No. 2, Feb. 1963.
2. Muto, Umemera, and Sonobe, "Study of the Overturning Vibration of Slender Structures", Proceedings, Second World Conference on Earthquake Engineering, Japan, 1960.
3. Hudson, D.E., "Response Spectrum Techniques in Engineering Seismology", Proceedings, World Conference on Earthquake Engineering, Berkeley, Calif., 1956.
4. Rosenblueth, E., "Some Applications of Probability Theory in Aseismic Design", Proceedings, World Conference on Earthquake Engineering, Berkeley, Calif., 1956.
5. Rosenblueth, E., "Probabilistic Design to Resist Earthquakes", Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM5, October, 1964.
6. Goldberg, J.E., Bogdanoff, J.L., and Bernard, M., "Response of a Simple Structure to a Random Earthquake Type Disturbance", Bulletin, Seismological Society of America, Vol. 51, No. 2, April, 1961.
7. Housner, G.W., "Characteristics of Strong-Motion Earthquakes", Bulletin, Seismological Society of America, Vol. 37, No. 1, January, 1947.

8. Housner, G.W., "Properties of Strong Motion Earthquakes", Bulletin, Seismological Society of America, Vol 45, Page 197, 1955.

9. Thomson, W.T., "Spectral Aspect of Earthquakes", Bulletin, Seismological Society of America, Vol. 49, January 1959.

10. Housner, G.W., and Jennings, P.C., "Generation of Artificial Earthquakes", Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM1, February, 1964.

11. Berg, G.V., and Housner, G.W., "Integrated Velocity and Displacement of Strong Earthquake Ground Motion", Bulletin, Seismological Society of America, Vol. 51, No. 2, April, 1961.

12. Bycroft, G.N., "White Noise Representation of Earthquakes", Journal of the Engineering Mechanics Division, ASCE, Vol. 86, No. EM2, April, 1960.

13. Rosenblueth, E., and Bustamente, J.I., "Distribution of Structural Response to Earthquakes", Journal of the Engineering Mechanics Division, ASCE, Vol. 88, No. EM3, June, 1962.

14. Shinozuka, M., "Probability of Structural Failure Under Random Loading", Journal of the Engineering Mechanics Division, ASCE, Vol. 90, No. EM5, October, 1964.

15. Gray, A.H., "First Passage Time in Random Vibrational System", *Journal of Applied Mechanics*, ASME, Vol. 88, March, 1966.

16. Roberts, J.B., "An Approach to the First Passage Problem in Random Vibration", *Journal of Sound and Vibration*. Vol. 8, No. 2, September, 1968.

17. May, K.F., "Pseudo-Random Number Generator", Program Library, Department of Computing Science, University of Alberta, Subroutine CS003A, September, 1967.

B30002